Distributed Solution to Economic Dispatch Problem of Energy-Water Nexus Systems

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Abstract

Electrical energy and water are two essential resources for the development of society. Furthermore, they are coupled in their production, distribution, and consumption. This complex relationship is often called the energy-water nexus. In this paper, we provide a distributed algorithm that can find the optimal solution of the economic dispatch problem of the energy-water nexus. Our convergence analysis shows that the trajectories converge to the optimal solution regardless of the initial allocation; therefore, we do not need to consider the initial procedure in detail. Moreover, the proposed algorithm does not require the plants to exchange the gradient information of cost functions with their neighbors. This means that the proposed algorithm can protect the plants' privacy. Finally, we provide a numerical example to demonstrate the performance and effectiveness of the proposed continuous-time algorithm for the energy-water nexus.

1 Introduction

Clean energy and water are two essential resources for the development of our society. Traditionally, the production of potable water and power generation are thought of as separable problems. However, with the development of technology, they are becoming increasingly correlated. For example, multi-stage flash (MSF) seawater desalination systems are usually coupled with power plants because they can use the wasted energy of used gas (which exits from gas turbine cycles) to produce potable water, while the gas turbine generator needs water to produce energy. With this scheme, it contributes to improving the fuel efficiency of the whole plant [1].

In recent years, a large number of researchers have investigated the energy-water nexus from various aspects. For example, Wanjiru and Xia [2] developed a model to save the energy and water by optimally controlling the lawn irrigation. Dubreuil et al. [3] implemented an energy optimization model with a dedicated water module to assess an optimal "water-energy" mix. Lubega and Farid [4] developed a quantitative, physics-based model of the energy-water nexus to optimize the energy and water systems from an engineering system perspective. Nanduri et al. [5] developed a competitive Markov decision process model for the energy-water-climate change nexus and the model was solved by a reinforcement learning algorithm. Tang et al. [6] established a multi-objective optimization model to study the room and realization path of both energy and water conservation under the prerequisite of stable economic development. Zhang and Vesselinov [7] developed an integrated model analysis framework and tool to help predict and satisfy water, energy, and food demands based on model inputs representing productions costs, socioeconomic demands, and environmental controls. Although these research efforts are helpful for solving the energy-water nexus issues, most work adopts centralized methods to handle the associated optimization problem. It means that these approaches require a control center to acquire and process all the data from the whole network. As the scale of network becomes larger and larger, the control center can't bear so much computation burden.

To overcome the weakness of centralized method, various distributed algorithms have been developed for large-scale network optimization problem. For example, Hu et al. [8] proposed a distributed adaptive droop control method for DC micro-grid to optimize power dispatch. Zhang et al. [9] designed a distributed method based on incremental cost for the conventional economic dispatch problem (EDP). Yang et al. [10] still adopted the equal incremental cost criterion to achieve the optimal dispatch, but the proposed algorithm can estimate the mismatch between demand and total power in a collective sense. Li et al. [11] firstly used the logarithmic barrier function to reformulate the economic dispatch problem with capacity constraints and then employed the consensus approach to design the distributed algorithm. Guo et al. [12] proposed a distributed algorithm on the basis of projected gradient and finite-time average consensus algorithms for smart grid systems. Xing et al. [13] used the method of bisection and a consensus-like iterative method to solve the economic dispatch problem (EDP) in a smart grid. Cherukuri and Cortés [14] proposed a distributed algorithm based on Laplacian-gradient dynamics for an EDP without and with capacities constraints. Then, Cherukuri and Cortés [15] extended the initial strategy and presented the algorithm which could converge to the optimal solution of the dispatch problem starting from any initial power allocation in a distributed manner. Yang et al. [16] proposed an algorithm which is capable of solving an EDP in a minimum number of time steps instead of asymptotically. Based on the alternating direction method of multipliers and finite-time average-consensus control strategy, Li et al. [17] proposed a distributed algorithm which can ensure that the generator constraints are satisfied during the whole computation process. Yi et al. [18] provided two classes of continuous-time algorithms to solve this resource allocation optimization problem in an initialization-free and scalable manner by adopting either projection or differentiated projection method. Deng et al. [19] developed a distributed algorithm for a resource allocation problem with local non-smooth cost function over weight-balanced digraphs. Zeng et al. [20] adopted a projected output feedback method to solve the resource allocation problem with uncertain parameters.

In this paper, we present a novel continuous-time distributed algorithm to solve the economic dispatch problem of the energy-water nexus with local inequality constraints in the framework of non-smooth analysis and algebraic graph theory. To the best of our knowledge, this is the first attempt to provide a quantitative solution for the economic dispatch problem of the energy-water nexus. Firstly, based on the exact penalty function method, we transform the original economic dispatch problem into an equivalent problem without local constraints. Then we propose a distributed algorithm to solve the equivalent problem in a distributed manner, which means that each plant or agent of the network only needs to obtain its neighbors' information. Therefore, the proposed algorithm can be effectively applied to the large-scale water and power network which consists of numerous plants. Compared with many existed algorithms which cannot get the correct optimal solution if the initial condition has an error, the proposed algorithm in this paper allows a group of plants to solve the economic dispatch problem for any initial value. Moreover, in our algorithm, we do not require the agents or plants to share their respective gradient information with their neighbors. In other words, the proposed algorithm can favorably protect plants' privacy.

The remainder of this paper consists of five sections. We provide some preliminaries about algebraic graph theory, non-smooth analysis, and set-valued dynamical systems in Section 2. The economic dispatch problem is formulated for the energy-water nexus and the exact penalty method is presented in Section 3. A distributed continuous-time algorithm is proposed and its convergence is proved in Section 4, while a simulation example is given in section 5. Finally, we provide our conclusions in section 6.

Notations: R and R^n represent the set of real numbers and real n-dimensional column vector, respectively; $\mathbf{1}_n$ (or $\mathbf{0}_n$) denotes an n-dimensional column vector whose all elements are 1 (or 0); for a vector or a matrix $\mathbf{x}$, $\mathbf{x}^T$ represents its transpose, and $\|\cdot\|$ represents the Euclidean norm of a vector or the corresponding induced norm of a matrix; $\inf(S)$ represents the greatest lower bound of the set S; $\otimes$ denotes the Kronecker product.

2 Preliminaries and Problem Formulation

In this section, some basic concepts about algebraic graph theory are firstly introduced. Then we review some notions from non-smooth analysis and differential inclusion. Finally, we formulate the economic dispatch problem of the energy-water nexus.

2.1 Graph Theory

Now we present some notions from algebraic graph theory [21]. A graph is a triplet $G = (V, E, A)$ where V denots the vertex set, $E \subseteq V \times V$ represents the edge set and A is the adjacency matrix. An edge from j to i, denoted by (i, j), means that agent i can receive information from agent j. An adjacency matrix is defined by $A = [a_{ij}] \in R^{N\times N}$, where $a_{ij} > 0$ if $(i, j) \in E$ and $a_{ij} = 0$, otherwise. A graph is undirected if $(i, j) \in E$ and $(j, i) \in E$ simultaneously. An undirected graph is connected if there is a path between any pair of vertexes. The Laplacian matrix is defined by $L = D-A$ where $D = \mathrm{diag}\{\mathrm{d_1}, \cdots, \mathrm{d_N}\}$ with $d_i = \sum_{j=1}^{N} a_{ij}$. For the Laplacian matrix L, at least one of the eigenvalues of L is zero and the rest of them have nonnegative real parts. If G is an undirected and connected graph, then 0 is a simple eigenvalue of L and the other eigenvalues are positive numbers. Additionally, the eigenvector corresponding to the eigenvalue 0 is given by $\nu \mathbf{1}_N$ for some constant ν. Throughout this paper, the following assumption is used for the graph $G = (V, E, A)$.

Assumption 1 *The graph $G = (V, E, A)$ is undirected and connected.*

2.2 Non-smooth Analysis and Differential Inclusion

We recall some notions from non-smooth analysis [22]. A function $f : R^n \to R$ is locally Lipschitz at $x \in R^n$ if there exist positive constants ϵ and δ, such that for any vectors $y, z \in B(x, \delta)$, one has $|f(y)-f(z)| \leq \epsilon\|y-z\|$. If f is Lipschitz near any point $x \in R^n$, then f is said to be locally Lipschitz in R^n. If the function f is locally Lipschitz in R^n, then f is differential almost everywhere (a.e.) in the sense of Lebesgue measure. The generalized directional derivative of f at x in the direction $\upsilon \in R^n$ is defined,

$$f^0(x; \upsilon) = \limsup_{y\to x, \xi\to 0^+} \frac{f(y + \xi\upsilon) - f(y)}{\xi}.$$

Furthermore, Clarke's generalized gradient of f at x is defined by

$$\partial f(x) = \{\zeta \in R^n : f^0(x; \upsilon) \geq \langle\zeta, \upsilon\rangle, \forall \upsilon \in R^n\}.$$

If $f : R^n \to R$ be a convex function, then one knows that $\partial f(x)$ is a nonempty, convex, compact set of R^n, and $\partial f(x)$ is upper semicontinuous at x.

A time-invariant differential inclusion is given by

$$\dot{x}(t) \in F(x(t)),\ x(0) = x_0,\ t \geq 0, \tag{1}$$

where F is a set-valued map from R^q to the compact convex subsets of R^q. A solution $x(t) : [0, \tau] \to R^q$ for $\tau > 0$ of the differential inclusion (1) is an absolutely continuous curve almost everywhere. A set M is said to be weakly invariant (resp., strongly invariant) with respect to (1), if for every $x_0 \in M$, M contains at least a solution (resp., all the solutions) of (1) starting from x_0. An equilibrium point of (1) is a point $x_e \in R^q$ with $0_q \in F(x_e)$. An equilibrium point $z \in R^q$ of (1) is Lyapunov stable if, for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that, for every initial condition $x(0) = x_0 \in B(z;\delta)$, every solution $x(t) \in B(z;\epsilon)$ for all $t \geq 0$.

Let $V : R^q \to R$ be a locally Lipschitz continuous function, and ∂V is the Clarke generalized gradient of $V(x)$ at x. The set-valued Lie derivative $L_F V : R^q \to B(R)$ of V with respect to the differential inclusion (1) is defined by $L_F V(x) \triangleq \{a \in R : p^T \upsilon = a, \upsilon \in F(x), p \in \partial V(x)\}$. In the case when $L_F V(x)$ is nonempty, we use $\max L_F V(x)$ to denote the largest element in $L_F V(x)$. If $\phi(\cdot)$ is a solution to (1) and $V : R^q \to R$ is locally Lipschitz and regular, then $\dot{V}(\phi(t))$ exists almost everywhere, and $\dot{V}(\phi(t)) \in L_F V(\phi(t))$ almost everywhere. In addition, if $V(\cdot)$ is continuous differentiable at x, then $L_F V(x) = \{\upsilon^T \nabla V(x),\ \upsilon \in F(x)\}$. Next, an invariance principle is presented for non-smooth regular functions [23].

Lemma 1 *For the differential inclusion (1), assume that F is upper semicontinuous and locally bounded, and $F(x)$ takes nonempty, compact and convex values. Let $V : R^q \to R$ be a locally Lipschitz and regular function, $S \subset R^q$ be compact and strongly invariant for (1), $\phi(\cdot)$ be a solution of (1),*

$$\mathsf{R} = \big\{x \in R^q | 0 \in L_F V(x)\big\},$$

and M be the largest weakly invariant subset of $\overline{\mathsf{R}} \cap S$, where $\overline{\mathsf{R}}$ is the closure of R. If $\max L_F V(x) \leq 0$ for all $x \in S$, then $d(\phi(t), M) \to 0$ as $t \to +\infty$ where $d(\phi(t), M) = \inf\big\{\|\phi(t) - y\|, y \in M\big\}$. For scalar s, $[s]^+ = s$ if $s > 0$, and $[s]^+ = 0$ otherwise.

2.3 Problem Formulation

We now present a mathematical model for co-optimization of power-water nexus. Let $x_{pi} \in R, x_{wj} \in R$ denote the power generated by the power plant i and the water produced by a water plant j respectively. Let $x_{cpk} \in R$ and $x_{cwk} \in R$ denote the power and water produced by a coproduction plant k. Let d_{pi}, d_{wj}, d_{cpk} and d_{cwk} denote the resource product demands. The following notations are introduced to vectorize the formulation: $\mathbf{x}_{pi} = [x_{pi}, 0]^T, \mathbf{x}_{wj} = [0, x_{wj}]^T, \mathbf{x}_{ck} = [x_{cpk}, x_{cwk}]^T, \mathbf{d}_{pi} = [d_{pi}, 0]^T, \mathbf{d}_{wj} = [0, d_{wj}]^T, \mathbf{d}_{ck} = [d_{cpk}, d_{cwk}]^T$. Then the co-optimization problem of power-water nexus is modeled as follows,

$$\begin{aligned}
&\min \sum_{i=1}^{N_p} f_{pi}(\mathbf{x}_{pi}) + \sum_{j=1}^{N_w} f_{wj}(\mathbf{x}_{wj}) + \sum_{k=1}^{N_c} f_{ck}(\mathbf{x}_{ck}),\\
s.t. &\sum_{i=1}^{N_p} \mathbf{x}_{pi} + \sum_{j=1}^{N_w} \mathbf{x}_{wj} + \sum_{k=1}^{N_c} \mathbf{x}_{ck} = \sum_{i=1}^{N_p} \mathbf{d}_{pi} + \sum_{j=1}^{N_w} \mathbf{d}_{wj} + \sum_{k=1}^{N_c} \mathbf{d}_{ck},\\
&\underline{\mathbf{x}}_{pi} \le \mathbf{x}_{pi} \le \overline{\mathbf{x}}_{pi} \qquad \text{for} \quad i = 1, \cdots, N_p,\\
&\underline{\mathbf{x}}_{wj} \le \mathbf{x}_{wj} \le \overline{\mathbf{x}}_{wj} \qquad \text{for} \quad j = 1, \cdots, N_w,\\
&\underline{\mathbf{x}}_{ck} \le \mathbf{x}_{ck} \le \overline{\mathbf{x}}_{ck} \qquad \text{for} \quad k = 1, \cdots, N_c.
\end{aligned} \tag{2}$$

where f_{pi}, f_{wj} and f_{ck} are respectively the scalar cost functions for the ith power production facility, the jth water production facility and the kth coproduction facility, N_p, N_w and N_c are the numbers of power, water and coproduction facilities, respectively, $\underline{\mathbf{x}}_{pi}, \underline{\mathbf{x}}_{wj}, \underline{\mathbf{x}}_{ck}, \overline{\mathbf{x}}_{pi}, \overline{\mathbf{x}}_{wj}$ and $\overline{\mathbf{x}}_{ck}$ are positive lower and upper bound vectors, respectively.

Remark 1 *The equality constraint is a global constraint which denotes the supply-demand balance while the inequality constraints represent the reasonable limits of plants' production capacity.*

Assumption 2 *The cost functions f_{pi}, f_{wj} and f_{ck} are convex and continuous differentiable.*

3 Distributed Algorithm for Economic Dispatch Problem

Before a distributed algorithm is proposed to solve the optimization problem (2), we need to transform the problem by using the exact penalty function method. We ignore the heterogeneity of the power, water and coproduction plants and define $\mathbf{x}_i \in R^2$ as the concatenation vector of $\mathbf{x}_{pi}, \mathbf{x}_{wj}$ and $\mathbf{x}_{ck}$, thus the index i ranges from 1 to $N = N_p + N_w + N_c$. Correspondingly, define $f_i(\mathbf{x}_i)$ as the concatenation vector functions of $f_{pi}(\mathbf{x}_{pi}), f_{wj}(\mathbf{x}_{wj})$ and $f_{ck}(\mathbf{x}_{ck})$ and $\mathbf{d}_i$ as the demand vector. Based on the optimization problem (2), a transformed model is given as follows,

$$\begin{aligned}
&\min f^{\theta}(\mathbf{x}) = \sum_{i=1}^{N} f_i^{\theta}(\mathbf{x}_i),\\
&s.t. \sum_{i=1}^{N} \mathbf{x}_i = \sum_{i=1}^{N} \mathbf{d}_i,
\end{aligned} \tag{3}$$

where $f_i^{\theta}(\mathbf{x}_i) = f_i(\mathbf{x}_i) + \frac{1}{\theta} \sum_{j=1}^{2} [(x_i^j - \overline{x}_i^j)^+ + (\underline{x}_i^j - x_i^j)^+]$, $x_i^j, \overline{x}_i^j$ and $\underline{x}_i^j$ denote the jth element of the state variable $\mathbf{x}_i$, the lower bound $\underline{\mathbf{x}}_i$ and the upper bound $\overline{\mathbf{x}}_i$, respectively. The parameter θ is a constant and satisfies the following lemma.

Lemma 2 *[14] The solutions to the optimization problems (2) and (3) coincide for* $\theta > 0$ *such that*

$$\theta < \frac{1}{2\max\limits_{\boldsymbol{x}\in\triangle_1}\|\partial f(\boldsymbol{x})\|_\infty},$$

where

$$\triangle_1 = \left\{\boldsymbol{x}\in R^{2N} | \sum_{i=1}^N \boldsymbol{x}_i = \sum_{i=1}^N \boldsymbol{d}_i \quad and \quad \underline{\boldsymbol{x}}_i \le \boldsymbol{x}_i \le \overline{\boldsymbol{x}}_i\right\}.$$

The generalized gradient $\frac{\partial f_i^\theta}{\partial x_i^j}$ is given as follows,

$$\frac{\partial f_i^\theta}{\partial x_i^j} = \begin{cases} \frac{\partial f_i}{\partial x_i^j} - \frac{1}{\theta}, & x_i^j \le \underline{x}_i^j, \\ [\frac{\partial f_i}{\partial x_i^j} - \frac{1}{\theta}, \frac{\partial f_i}{\partial x_i^j}], & x_i^j = \underline{x}_i^j, \\ \frac{\partial f_i}{\partial x_i^j}, & \underline{x}_i^j \le x_i^j \le \bar{x}_i^j, \\ [\frac{\partial f_i}{\partial x_i^j}, \frac{\partial f_i}{\partial x_i^j} + \frac{1}{\theta}], & x_i^j = \bar{x}_i^j, \\ \frac{\partial f_i}{\partial x_i^j} + \frac{1}{\theta}, & x_i^j \ge \bar{x}_i^j. \end{cases}$$

Now, a distributed algorithm is provided to solve the problem (3)

$$\begin{cases} \dot{\mathbf{x}}_i \in -\partial f_i^\theta(\mathbf{x}_i) + \lambda_i, \\ \dot{\lambda}_i = -\sum_{j=1}^N(\lambda_i - \lambda_j) - \sum_{j=1}^N(\mathbf{z}_i - \mathbf{z}_j) + (\mathbf{d}_i - \mathbf{x}_i), \\ \dot{\mathbf{z}}_i = \sum_{j=1}^N(\lambda_i - \lambda_j), \end{cases} \tag{4}$$

where $\lambda_i = (\lambda_i^1, \lambda_i^2)^T$, and $\mathbf{z}_i = (z_i^1, z_i^2)^T$ for $i = 1, 2, \cdots, N$. Equivalently, the algorithm (4) can be rewritten as the following vector form,

$$\begin{cases} \dot{\mathbf{x}} \in -\partial f^\theta(\mathbf{x}) + \lambda, \\ \dot{\lambda} = -(L\otimes I_2)\lambda - (L\otimes I_2)\mathbf{z} + (\mathbf{d} - \mathbf{x}), \\ \dot{\mathbf{z}} = (L\otimes I_2)\lambda, \end{cases} \tag{5}$$

where $\lambda = [\lambda_1^T, \lambda_2^T, \cdots, \lambda_N^T]^T$, $\mathbf{z} = [\mathbf{z}_1^T, \mathbf{z}_2^T, \cdots, \mathbf{z}_N^T]^T$ and $\mathbf{d} = [\mathbf{d}_1^T, \mathbf{d}_2^T, \cdots, \mathbf{d}_N^T]^T$.

Theorem 1 *Under Assumption 1 and Assumption 2, if* $(\boldsymbol{x}^*, \lambda^*, \boldsymbol{z}^*)$ *is an equilibrium point of the distributed algorithm (5), then* $\boldsymbol{x}^*$ *is an optimal solution of the problem (3).*

Proof. Since $(\mathbf{x}^*, \lambda^*, \mathbf{z}^*)$ is an equilibrium point of the algorithm (5), then one has,

$$\begin{cases} 0 \in -\partial f^\theta(\mathbf{x}^*) + \lambda^*, \\ 0 = -(L\otimes I_m)\lambda^* - (L\otimes I_m)\mathbf{z}^* + (\mathbf{d} - \mathbf{x}^*), \\ 0 = (L\otimes I_m)\lambda^*. \end{cases}$$

Because the graph G is undirected and connected, we have $\mathbf{1}_N^T L = 0$. Furthermore, one has

$$\begin{aligned}
&(\mathbf{1}_N^T \otimes I_2)[-(L \otimes I_2)\lambda^* - (L \otimes I_2)\mathbf{z}^* + (\mathbf{d} - \mathbf{x}^*)] \\
=& - [(\mathbf{1}_N^T L) \otimes I_2]\lambda^* - [(\mathbf{1}_N^T L) \otimes I_2]\mathbf{z}^* + (\mathbf{1}_N^T \otimes I_2)(\mathbf{d} - \mathbf{x}^*) \\
=&(\mathbf{1}_N^T \otimes I_2)(\mathbf{d} - \mathbf{x}^*) = \mathbf{0}_2.
\end{aligned}$$

Thus one has $\sum_{i=1}^N \mathbf{x}_i^* = \sum_{i=1}^N \mathbf{d}_i$. Additionally, since $(L \otimes I_2)\lambda^* = \mathbf{0}_2$, one has $\lambda_1^* = \lambda_2^* \cdots = \lambda_N^* = \mu \in R^2$. Therefore, the equilibrium point $(\mathbf{x}^*, \lambda^*, \mathbf{z}^*)$ of the algorithm (5) satisfies

$$\begin{aligned}
&\mathbf{0}_{N\times 2} \in -\partial f^\theta(\mathbf{x}^*) + \mathbf{1}_N \otimes \mu, \\
&\sum_{i=1}^N \mathbf{x}_i^* = \sum_{i=1}^N \mathbf{d}_i,
\end{aligned}$$

which is exactly the optimal condition for problem (3). Thus, the equilibrium point $\mathbf{x}^*$ is the optimal solution of the problem (3). The proof is completed.

Remark 2 *From Theorem 1, it is sufficient to show that the algorithm (5) can converge to the optimal solution* $\boldsymbol{x}^*$ *if the algorithm (5) converges to the equilibrium point* $(\boldsymbol{x}^*, \lambda^*, \boldsymbol{z}^*)$. *Next, we show that the algorithm (5) converges to the equilibrium point.*

Theorem 2 *For any initial point* $(\boldsymbol{x}(0), \lambda(0), \boldsymbol{z}(0))$, *the solutions of the algorithm (5) converge to the equilibrium point* $(\boldsymbol{x}^*, \lambda^*, \boldsymbol{z}^*)$ *under Assumption 1 and Assumption 2.*

Proof. Define an energy function

$$V(\mathbf{x}, \lambda, \mathbf{z}) = \frac{1}{2}\|\mathbf{x} - \mathbf{x}^*\|^2 + \frac{1}{2}\|\lambda - \lambda^*\|^2 + \frac{1}{2}\|\mathbf{z} - \mathbf{z}^*\|^2. \tag{6}$$

It is noted that, when $\mathbf{0}_{N\times 2} \in -\partial f^\theta(\mathbf{x}^*) + \lambda^*$, there exists a vector $g^* \in \partial f(\mathbf{x}^*)$ such that $\mathbf{0}_{N\times 2} = -g^* + \lambda^*$. Thus, one has

$$\begin{aligned}
L_F V(\mathbf{x}, \lambda, \mathbf{z}) =&(\mathbf{x} - \mathbf{x}^*)^T(-\partial f^\theta(\mathbf{x}) + \lambda) + (\lambda - \lambda^*)^T \\
&[-(L \otimes I_2)\lambda - (L \otimes I_2)\mathbf{z} + \mathbf{d} - \mathbf{x}] + (\mathbf{z} - \mathbf{z}^*)^T(L \otimes I_2)\lambda \\
=& - (\mathbf{x} - \mathbf{x}^*)^T(g - g^*) - (\lambda - \lambda^*)^T(L \otimes I_2)(\lambda - \lambda^*),
\end{aligned}$$

for any $g \in \partial f^\theta(\mathbf{x})$. Since $f^\theta(\mathbf{x})$ is convex and the Laplacian matrix is positive semidefinite, one has

$$-(\mathbf{x} - \mathbf{x}^*)^T(g - g^*) - (\lambda - \lambda^*)^T(L \otimes I_2)(\lambda - \lambda^*) \leq 0$$

Thus, $\max L_F V(\mathbf{x}, \lambda, \mathbf{z}) \leq 0$ and $V(\mathbf{x}(t), \lambda(t), \mathbf{z}(t))$ is non-increasing. It follows that $V(\mathbf{x}(t), \lambda(t), \mathbf{z}(t)) \leq V(\mathbf{x}(0), \lambda(0), \mathbf{z}(0))$. Furthermore, the solution $(\mathbf{x}(t), \lambda(t), \mathbf{z}(t))$ is bounded. Then we know that

$$F(\mathbf{x}, \lambda, \mathbf{z}) = \begin{pmatrix} -\partial f^\theta(\mathbf{x}) + \lambda \\ -(L \otimes I_2)\lambda - (L \otimes I_2)\mathbf{z} + (\mathbf{d} - \mathbf{x}) \\ (L \otimes I_2)\lambda \end{pmatrix}$$

is upper semicontinuous and locally bounded. At the same time, the energy function $V(\mathbf{x}, \lambda, \mathbf{z})$ is a locally Lipschitz and regular function. Let $S = \{(\mathbf{x}, \lambda, \mathbf{z}) | 0 \in L_F V(\mathbf{x}, \lambda, \mathbf{z})\}$. Since $L_F V(\mathbf{x}, \lambda, \mathbf{z}) \leq 0$, S is a compact and strongly invariant set for the algorithm (5). Thus, according to Lemma 1, the solution of the algorithm (5) starting from any initial value converges to the largest weakly positively invariant set M.

Next, we will show that any solution $(\overline{\mathbf{x}}, \overline{\lambda}, \overline{\mathbf{z}}) \in M$ is an optimal solution of the algorithm (5). Since $0 \in L_F V(\overline{\mathbf{x}}, \overline{\lambda}, \overline{\mathbf{z}})$, then one has

$$\begin{cases} (\overline{\lambda} - \lambda^*)^T (L \otimes I_2)(\overline{\lambda} - \lambda^*) = 0, \\ (\overline{\mathbf{x}} - \mathbf{x}^*)^T (g - g^*) = 0. \end{cases}$$

Due to $(L \otimes I_m)\lambda^* = \mathbf{0}$, one has $(L \otimes I_m)\overline{\lambda} = \mathbf{0}$. At the same time, it is noted that the Hessian matrix $H(x)$ of the function $f(x)$ is positive definite, one has

$$(\overline{\mathbf{x}} - \mathbf{x}^*)^T (g - g^*) = (\overline{\mathbf{x}} - \mathbf{x}^*)^T H(\tau \overline{\mathbf{x}} + (1 - \tau)\mathbf{x}^*)(\overline{\mathbf{x}} - \mathbf{x}^*) = 0,$$

where $0 \leq \tau \leq 1$. Hence, we have $\overline{\mathbf{x}} = \mathbf{x}^*$, which implies that $\overline{\mathbf{x}}$ is an optimal solution of the problem (3). Due to the arbitrariness of $(\overline{\mathbf{x}}, \overline{\lambda}, \overline{\mathbf{z}})$ in M, we can conclude that the equilibrium points in M are all the optimal solutions of the problem (3).

Finally, we prove that the solution $\mathbf{x}(t)$ to the algorithm (5) will converge to the largest weakly positively invariant set M. Define $\phi(t) = (\mathbf{x}(t), \lambda(t), \mathbf{z}(t))$. Since $\phi(t)$ is bounded, according to Bolzano-Weierstrass theorem [24], there exists $\widehat{\phi} = (\widehat{\mathbf{x}}, \widehat{\lambda}, \widehat{\mathbf{z}})$ and $\{t_k, k = 1, 2, \cdots\}$ such that $\phi(t_k) = (\mathbf{x}(t_k), \lambda(t_k), \mathbf{z}(t_k)) \to (\widehat{\mathbf{x}}, \widehat{\lambda}, \widehat{\mathbf{z}})$ as $k \to +\infty$. Since $\mathrm{dist}(\phi(t), M) \to 0$, then one has $\widehat{\phi} \in M$ and thus $\widehat{\mathbf{x}}$ is an optimal solution of the problem (3). Consider a new Lyapunov function $\bar{V}(\mathbf{x}(t), \lambda(t), \mathbf{z}(t))$ defined as $V(\mathbf{x}(t), \lambda(t), \mathbf{z}(t))$ by replacing $(\mathbf{x}^*, \lambda^*, \mathbf{z}^*)$ in the equation (6) with $(\widehat{\mathbf{x}}, \widehat{\lambda}, \widehat{\mathbf{z}})$. By similar discussion as above for $V(\mathbf{x}(t), \lambda(t), \mathbf{z}(t))$, one has $\max L_F \bar{V}(\mathbf{x}, \lambda, \mathbf{z}) \leq 0$. Due to the continuity of $\bar{V}$, for any $\epsilon > 0$, there exists $\omega > 0$ such that $\bar{V}(\mathbf{x}, \lambda, \mathbf{z}) \leq \epsilon$ when $\|\phi - \widehat{\phi}\| \leq \omega$. Since $\bar{V}$ is monotonically nonincreasing on interval $[0, +\infty)$, there exists a positive integer T such that when $t \geq t_T$, there is

$$\frac{1}{2}\|\mathbf{x}(t) - \widehat{\mathbf{x}}\|^2 \leq \bar{V}(\mathbf{x}(t), \lambda(t), \mathbf{z}(t)) \leq \bar{V}(\mathbf{x}(t_T), \lambda(t_T), \mathbf{z}(t_T)) < \epsilon \tag{7}$$

which implies $\lim_{t \to +\infty} \mathbf{x}(t) = \widehat{\mathbf{x}}$. Furthermore, since $\bar{V}$ is radially unbounded with respect to $\mathbf{x}(t)$, thus the solution $\mathbf{x}(t)$ to the algorithm (5) globally converges to an optimal solution of the problem (3).

Remark 3 *It is noted that the algorithm (4) is a differential inclusion and implemented by using only the plant's neighbors information. The gradient information is not shared in the neighborhood and thus the proposed algorithm can favorably protect the agents privacy.*

4 Simulation

In this section, we apply the algorithm (5) into a energy-water nexus system. It is assumed that the energy-water nexus system consists of 2 power plants, 2 water plants and 2 co-production facilities. The communication topology of the system is illustrated in Fig 1. The cost functions

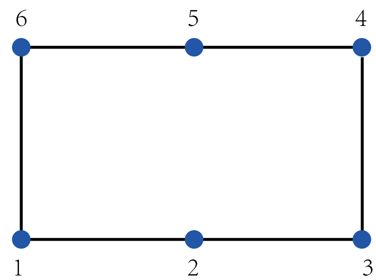

Figure 1: The communication network

Table 1: Plant and cost data

Plant type	Index	Max power capacity (MW)	Min power capacity (MW)	Max water capacity (m^3/h)	Min water capacity (m^3/h)
power	i_1	200	50	0	0
power	i_2	80	20	0	0
water	j_1	0	0	50	15
water	j_2	0	0	35	10
coproduction	k_1	80	16	40	10
coproduction	k_2	60	10	75	20

The cost parameters of power plants

A_p	B_p	$\mathcal{K}_p$
0.00375	2	5
0.00175	1.75	10

The cost parameters of water plants

A_w	B_w	$\mathcal{K}_w$
0.00625	1.0	7
0.00834	3.25	5.8

The cost parameters of coproduction plants

A_{c11}	A_{c12}	A_{c21}	A_{c22}	B_{c1}	B_{c2}	$\mathcal{K}_c$
0.007433	0.03546	0	0.007093	1.106	4.426	57
0.07881	0.06305	0	0.01261	1.475	5.901	57

f_{pi}, f_{wj}, f_{ck} of the plants are quadratic functions, i.e.,

$$f_{pi} = x_{pi}^T A_{pi} x_{pi} + B_{pi} x_{pi} + C_{pi},$$

$$f_{wj} = x_{wj}^T A_{wj} x_{wj} + B_{wj} x_{wj} + C_{wj},$$

$$f_{ck} = x_{ck}^T A_{ck} x_{ck} + B_{ck} x_{ck} + C_{ck}.$$

The parameters of the plants are given in Table 1. The demand vector is given by $\mathbf{d} = [90, 0, 20, 0, 0, 90, 0, 10, 50, 20, 45, 40]^T \in R^{12}$. The parameter θ in the penalty function $f^{\theta}(\mathbf{x})$

is chosen as 0.1019. The initial condition of the algorithm (5) can be randomly given. It is noted that the state $\mathbf{x}_i \in R^2$ of each agents has two components (i.e., power and water), thus we respectively give the evolution of the power and water states of the six plants under the algorithm (5), as shown in Fig. (2) and Fig. (3). In the figures, the black dashed lines denote the

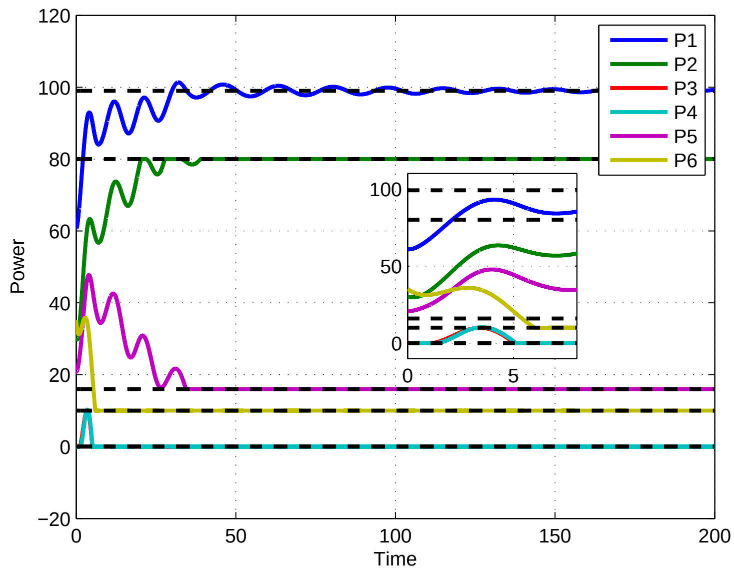

Figure 2: Evolution of power allocation

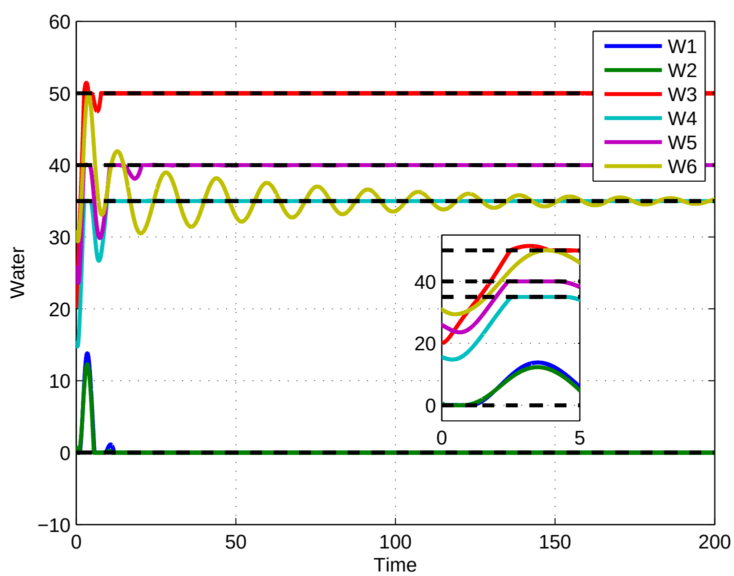

Figure 3: Evolution of water allocation

optimal solution of the algorithm (5), which is given by $x^*_{p1} = 99, x^*_{p2} = 80, x^*_{cp1} = 16, x^*_{cp2} = 10$

for power and $x_{w1}^* = 50, x_{w2}^* = 35, x_{cw1}^* = 40, x_{cw2}^* = 35$ for water. It can be found that the trajectories of the decision variables about power and water asymptotically converge to the optimal values. Additionally, Fig. 4 and Fig. 5 show that the proposed algorithm can keep the balance of supply and demand of power and water, respectively.

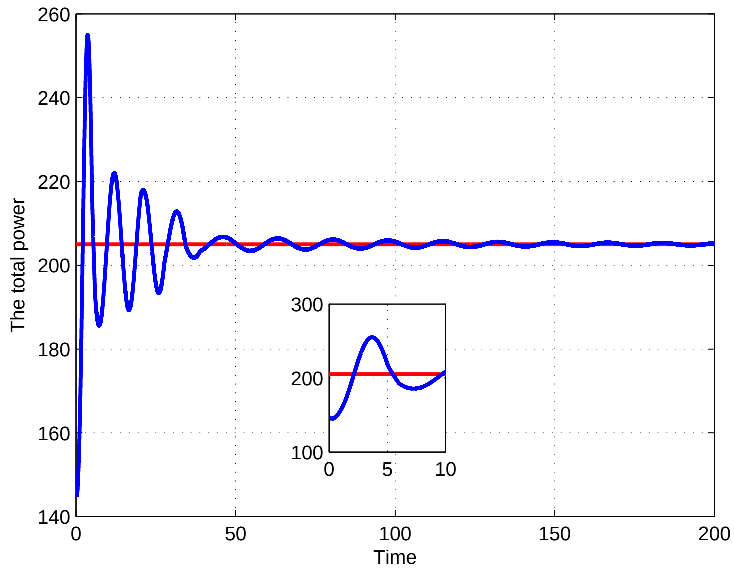

Figure 4: Total mismatch of power

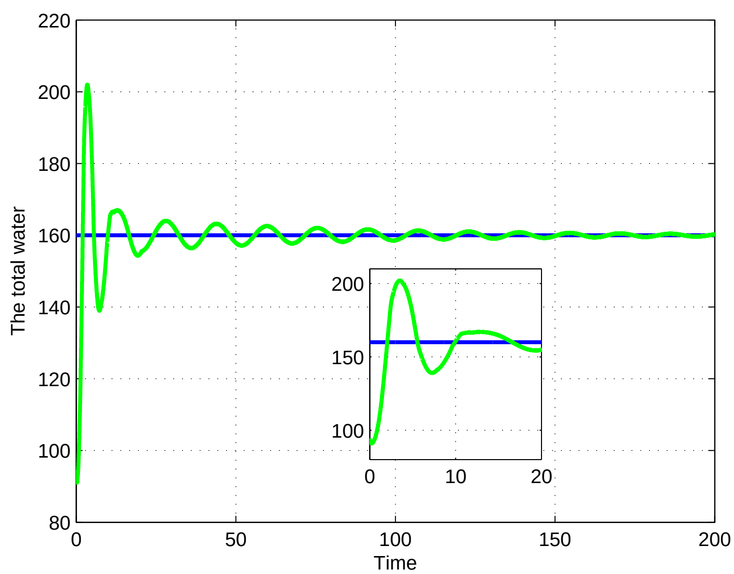

Figure 5: Total mismatch of water

4 Conclusion

In this paper, we have designed a distributed continuous-time algorithm which allows a group of power, water,and co-production plants to solve the economic dispatch problem with local inequality constraints starting from any initial allocation. To accomplish this, we have transformed the original problem by using the exact penalty function method into an equivalent problem without local inequality constraints. Then a distributed algorithm has been proposed to solve the equivalent optimization problem. A theoretical analysis has been presented for the proposed algorithm with the help of algebraic graph theory, Lyapunove function method and non-smooth analysis. The simulation result indicated that the algorithm is effective and it will save plenty of production cost for the economic dispatch problem of energy-water nexus. In the future, we will further investigate the economic dispatch problem of the energy-water nexus system which not only includes the electricity-storage devices but also the water storage facilities. Moreover, we will consider the power-water production ratio for the coproduction plants in the co-optimization model such that the model is more practical

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