Zero-Sum Repeated Games: Basic Results and New Advances

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Here we will concentrate on the zero-sum case.

Recall the simplest case where the game is defined by a $I \times J$ real valued matrix. A_{ij} decribes the gain of player 1 (and the loss of player 2) if player 1 (resp. 2) chooses the move i (resp. j).

Let $\Delta(I) = X$, the simplex on I, denote the set of mixed moves (and similarly $\Delta(J) = y$) and extend A to $X \times Y$ in a bilinear way.

The celebrated minmax theorem (von Neumann, 1928) states that there exists a triple $(v, x, y) \in R \times X \times Y$ with

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More generally one considers a zerosum game defined by a map $F : X \times Y \longrightarrow \boldsymbol{R}$.
Player 1 can obtain: $\underline{v} = \sup_X \inf_Y F(x,y)$ and Player 2 : $\overline{v} = \inf_Y \sup_X F(x,y)$.
The game has a value when $\underline{v} = \overline{v}$.
A general result is due to Sion (1958) and involves geometric conditions:
X and Y convex, F quasiconcave in x and quasiconvex in y and topological conditions X or Y compact, F upper semicontinuous in x and lower semicontinuous in y.

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We now consider the case where the stage game belongs to a family G^k, $k \in K$ of two-person zero-sum games played on strategy sets $I \times J$. The two basic classes of repeated games that have been extensively studied are stochastic games and incomplete information games.

In the first class, the parameter k is a publicly known variable, controlled by the players. It evolves along a play and its value at stage $n+1$, called the state k_{n+1}, is stationary random function of the triple (i_n, j_n, k_n) which are the moves, respectively the state, at stage n. At each stage both players share the same information and in particular they know the current state.

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However this one is changing and the issue for the players at stage n is to control both the current payoff g_n (induced by (i_n, j_n, k_n)) and the next state k_{n+1}.

In the second class, the parameter k is chosen once for all and kept fixed during the play. However, at least one player does not have full information about it. In this framework, the issue is the tradeoff between using the information (this increases the set of strategies which are the probabilty distribution of the moves, given the past) and revealing it.

We will see later on that these two models apparently very different - state known and changing versus state unkown and fixed- are in fact particular incarnations of a much more general model and that they share a lot of properties.

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A play of the game generates a sequence of stagepayoffs $\{g_n\}$. There are several ways of comparing outcomes in such a framework.

We first introduce the **compact case**.

For every probability distribution μ on the integers $n \geq 1$ ($\mu(n) \geq 0, \sum_n \mu(n) = 1$), one can define a game $\Gamma[\mu]$ with evaluation $\sum_n g_n \mu(n)$.

Under standard assumptions on the basic data, the natural topology on plays induces a game with compact strategy spaces and continuous payoff function, hence the value $v[\mu]$ will exist by Sion's theorem.

The **asymptotic approach** studies the family of such games as the expected lenght (the mean of μ) goes to $+\infty$.

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Two typical examples correspond to:
1) the **finite n-stage** game Γ_n with outcome given by the average of the first n payoffs:

$$\gamma_n = \frac{1}{n}\sum_{t=1}^{n} g_t$$

2) the λ-**discounted game** Γ_λ with outcome equal to the discounted sum of the payoffs:

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The values of these games are denoted by v_n and v_λ respectively. The study of their asymptotic behavior, as n goes to ∞ or λ goes to 0 is one example of the study of the **asymptotic game**.

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Extensions consider games with random duration process where the weight $\mu(n)$ is a random variable which law depends upon the previous path on the random duration tree (Neyman, 2003, Neyman and Sorin, 2010).
Note that the knowledge of the duration (i.e. the evaluation process) is crucial in the definition of the strategies.

An alternative analysis, called the **uniform approach**, considers the whole family of “long games”.
It does not specify outcome in some infinitely repeated game like $\liminf \frac{1}{n}\sum_{t=1}^{n} g_t$ or a measurable function defined on plays (see Maitra and Sudderth, 1998), but look for strategies exhibiting asymptotic uniform properties in the following sense: they are optimal in any sufficiently long game.

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Explicitely, $\underline{v}$ is the **maxmin** if the two following conditions are satisfied:

- Player 1 can **guarantee** $\underline{v}$: for any $\varepsilon > 0$, there exists a strategy σ of Player 1 and an integer N such that for any $n \geq N$ and any strategy τ of Player 2:

$$\gamma_n(\sigma, \tau) \geq \underline{v} - \varepsilon$$

where $\gamma_n(\sigma, \tau)$ is the expectation of γ_n under the pair of strategies (σ, τ).

- Player 2 can **defend** $\underline{v}$: for any $\varepsilon > 0$ and any strategy σ of Player 1, there exist an integer N and a strategy τ of Player 2 such that for all $n \geq N$:

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A dual definition holds for the **minmax** $\overline{v}$.
Whenever $\underline{v} = \overline{v}$, the game has a **uniform value**, denoted by v_∞.
Remark that the existence of v_∞ implies:

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3.1 Stochastic games and Shapley operator

a) Consider a stochastic game with state space K, action spaces I and J (all finite) and real payoff function g defined on $K \times I \times J$. The initial state k_1 is announced to both players. In addition at each stage $t+1$, the transition probability $Q(\cdot|k_t, i_t, j_t)$ defines the law of the next state k_{t+1}. Assume that the past history $(k_1, i_1, j_1, \cdots, k_t, i_t, j_t, k_{t+1})$ is known by the players at each stage $t+1$. Introduce the set of mixed (random) moves $X = \Delta(I)$ and $Y = \Delta(J)$ and extend by bilinearity g and Q to $X \times Y$.

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b) The Shapley operator Ψ acts on the set $\mathcal{F}$ of real functions f on K as follows. First define an auxiliary one shot game on $X \times Y$ by the payoff

$$\Psi_{xy}(f)(k) = g(k,x,y) + \int_K f(k')Q(k'|k,x,y) \qquad (1)$$

and then let

$$\Psi(f)(k) = \mathtt{val}_{X \times Y} \Psi_{xy}(f)(k) \qquad (2)$$

where $\mathtt{val}_{X \times Y}$ stands for the value operator:

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It is natural to introduce the ε-weighted operator corresponding to the ε-weighted auxiliary game

$$\Phi_{xy}(\varepsilon, f)(k) = \varepsilon g(k, x, y) + (1 - \varepsilon) \int_K f(k') Q(k' | k, x, y) \tag{3}$$

and then

$$\Phi(\varepsilon, f)(k) = \mathtt{val}_{X \times Y} \Phi_{xy}(\varepsilon, f)(k) \tag{4}$$

which is related to the initial Shapley operator by :

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The discounted stochastic game Γ_λ has a value which is the unique solution of

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By using the proposed strategy player 1 obtains

$$E[\lambda g_n + (1-\lambda) w(k_{n+1})] \geq w(k_n)$$

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Ψ determines the family of values through:

$$n v_n = \Psi^n(0), \qquad \frac{v_\lambda}{\lambda} = \Psi((1-\lambda)\frac{v_\lambda}{\lambda}). \tag{6}$$

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$$\begin{array}{c|c|c|} & \alpha & \beta \\ \hline a & 1^* & 0^* \\ \hline b & 0 & 1 \\ \hline \end{array}$$

This corresponds to a stochastic game where, as soon as Player 1 plays a, the game reaches an absorbing state with a constant payoff corresponding to the entry played at that stage. Both the n-stage value v_n and the λ-discounted value v_λ are equal to 1/2 and are also independent of the additional information transmitted along the play to the players.

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However the corresponding optimal strategies (play a with probability $1/(n+1)$ or $\lambda/(1+\lambda)$) do not exhibit good asymptotic properties.

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v_∞ exists.

The proof relies on the construction of a ε-optimal strategy of player 1. Define $L_n = n(\gamma_n - 1/2)$ and play a at stage $n+1$ with probability $\max\{M, L_n\}^2$ where M is a large parameter adjusted to ε.

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f) The most important result in the field is the next general existence result

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The proof uses the bounded variation property of the family v_λ to define an evaluation function $\bar{\lambda}$ that will adjust the "horizon" to the current performance. At stage $n+1$ compute some statistics m_{n+1} from the past history of payoffs and states (roughly equal to $m_n + g_n - v_{\lambda_n}(k_n)$) and play (once) optimally in the auxiliary game $\Phi_{xy}(\lambda_{n+1}, v_{\lambda_{n+1}})(k_{n+1})$ where the discount factor is $\lambda_{n+1} = \bar{\lambda}(m_{n+1})$.

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3.2 Games with incomplete information

We consider here the simplest framemork: independent case and standard signalling.

The repeated game $\Gamma(p,q)$ is defined by a family $\{G^{k\ell}\}, k \in K, \ell \in L$, of two-person zero-sum games played on $I \times J$. All the sets are finite.

In addition two probabilities, p on K and q on L are given, according to which a couple (k,ℓ) is selected and player 1 (resp. 2) is informed upon k (resp. ℓ). Then player 1 (resp. 2) chooses, at each stage n, a move $i_n \in I$ (resp. $j_n \in J$) and the only information transmitted to both is the couple (i_n, j_n). The stage payoff, $G^{k\ell}_{i_n j_n}$, is usually unknown.

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Denote by $D(p, q)$ the average game $\sum_{k,\ell} p^k q^\ell G^{k\ell}$ and by $u(p, q)$ its value. Note that this corresponds to the game where none of the players is informed - or is using his information.

a) Lack of information on one side

This is the case where $\#L = 1$ hence we drop this index. Player 1 knows the true game while player 2 only knows the initial prior p.

Theorem

Aumann and Maschler (1966)
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First the informed player can generate any martingale with respect to his private information (splitting lemma); since player 1 can obtain u by not using his information, he can also obtain Cav u.

The dual inequality is obtained as follows: knowing the strategy σ of player 1, player 2 can compute stage after stage a new posterior probability on K. This defines a martingale p_n and its variation at each stage is proportional to the gain that player 1 can achieve by using his information. The total L^1 variation of a bounded martingale on n stages being at most of the order of $\sqrt{n}$ this gives a speed of convergence of $1/\sqrt{n}$.

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For player 2 a concatenation of optimal strategies in longer and longer games $\Gamma_n(p)$ allows to guarantee Cav u.
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The fact that player 2 can defend this amount relies on a strategy τ exhausting from the strategy σ of player 1 a maximal amount of information. From some stage on, player 1 is not revealing any information anymore and we are back to a situation with incomplete information on one side.

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Note the 2 points:

first, the analysis is disymmetric, since we defined properties in terms of “best response”. In fact the maxmin and minmax may differ: there is no uniform value.

second, the time needed to exhaust information is not bounded hence this reply is not adapted to the “compact case”.

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In fact the asymptotic analysis leads to the following result:

Theorem

Mertens and Zamir (1971)
$\lim v_\lambda$ *and* $\lim v_n$ *exist, are equal and are the unique solution of the set of functional equations*

$$v = \text{Cav}_p \min\{u, v\} \qquad v = \text{Vex}_q \min\{u, v\}$$

The basic step is the following result:
If player 1 can obtain asymptotically w, he can also obtain asymptotically $\text{Vex}max\{u, w\}$: follow at each stage n an optimal strategy for $u(p_n, q_n)$ as long as this quantity is more than $w(p_n, q_n)$ and switch to an optimal for w otherwise.

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4.1 General model

Let M be a parameter space and g be a function g from $I \times J \times M$ to $\mathbb{R}$: for each $m \in M$. This defines a two person zero-sum game with action spaces I and J for Player 1 and 2 respectively and payoff function g.

(Again to simplify the presentation we will consider the case where all sets are finite, avoiding in particular measurability issues).

The initial parameter m_1 is chosen at random and the players receive some initial information about it, say a_1 (resp. b_1) for player 1 (resp. player 2). This choice is performed according to some probability π on $M \times A \times B$, where A and B are the signal sets of each player.

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Then at each stage n, player 1 (resp. 2) chooses a move $i_n \in I$ (resp. $j_n \in J$). In addition, after each stage the players obtain some further information about the previous choice of actions and both the previous and the current values of the parameter. This is represented by a map Q from $M \times I \times J$ to probabilities on $M \times A \times B$. At stage n given the state m_n and the moves (i_n, j_n), a triple $(m_{n+1}, a_{n+1}, b_{n+1})$ is choosen at random according to the distribution $Q(m_n, i_n, j_n)$. The new parameter is m_{n+1}, and the signal a_{n+1} (resp. b_{n+1}) is transmitted to player 1 (resp. player 2).

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A play of the game is thus a sequence $m_1, a_1, b_1, i_1, j_1, m_2, a_2, b_2, i_2, j_2, \ldots$ while the information of Player 1 before his play at stage n is a 1-private history of the form $(a_1, i_1, a_2, i_2, \ldots, a_n)$ and similarly for Player 2.

The corresponding sequence of payoffs is $g_1, g_2, \ldots$ with $g_n = g(i_n, j_n, m_n)$. (Note that it is not known to the players except if included in the signals).

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A strategy σ for player 1 is a map from 1-private histories to $\Delta(I)$, the space of probabilities on the set I of actions: it defines the probability disribution of the stage move as a function of the past known to player 1; τ is defined similarly for player 2. Such a couple (σ, τ) induces, together with the components of the game, π and Q, a probability distribution on plays, hence on the sequence of payoffs.

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4.2 Games with incomplete information

The standard model presented above corresponds to the case $m_1 = (k, \ell)$, $\pi = p \times q$, initial information $a_1 = k$, $b_1 = \ell$, then $m_n = m_1$ is constant and the signals reveals the moves $(a_{n+1} \ = b_{n+1} = (i_n, j_n))$.

We discuss here extensions where the signalling structure is more general.

a) Signals: one side

The basic remark (Aumann and Maschler,1968) is that the 2 notions of using the information (i.e. the distribution of the move i is a fonction $x^k \in X$ of the parameter) and revealing it are no longer equivalent.

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From the previous analysis relying on martingale arguments it follows that the right notion is ”revelation”.

Define the set NR of non revealing strategies as the vectors $\{x^k\}$ of mixed moves such that all components induce the same signals to player 2 whatever being his move.

$D(p)$ is the one-shot game where player 1 plays in NR and player 2 plays in Y and u is its value.Then one still has

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b) Signals: both sides

When the signalling structure is independent of the state - meaning that the transmission of information is only trough the signals induced by the moves an analysis similar to the one done in the case of lack of information on one side is available. Player 1 has a set NR^1 of non revealing strategies (that depends only upon the signals to player 2) and similarly for player 2. Then one defines the non revealing game $D(p,q)$ played on $NR^1 \times NR^2$ with payoff

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$\underline{v}$ and $\overline{v}$ exists and satisfy

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The situation is however drastically when the signalling structure may depend upon the state: revelation may occur even if one player is not using his information.

The field is still to explore there but significant advances have been achieved like the construction of an auxiliary "normal form" game for the uniform approach (Mertens and Zamir, 1976b and Waternaux, 1983) and a "double scale" approach to the limit game (Sorin 1985b).

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The relation with stochastic games can be seen on the following signalling matrices corresponding to a 2×2 game with 2×2 states (Sorin, 1989)

T	L
P	Q

H^{11}

T	R
P	Q

H^{12}

B	L
P	Q

H^{21}

B	R
P	Q

H^{22}

The analysis a of such games leads to the study of the whole class of stochastic games with incomplete information on one side. Only partial results are still available (Sorin, 1984, 1985a, Rosenberg and Vieille, 2000) but important differences appear with the previous sub-classes:
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4.3 Stochastic games

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a) Symmetric case

Consider the case where the state may not be known by the players but their information is symmetric (hence include their moves). The players may collect information upon the current state trough their moves and the natural state space is their probability on M.

Theorem

In the symmetric case the repeated game Γ has a value.

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b) Stochastic games with signals

We assume here that the signal to each player reveals the current stage but not necessarily the previous move of the opponent. By the recursive formula for v_λ and v_n these quantities are the same; however for example in the Big Match, when Player 1 has no information on Player 2's moves the $\max\min$ is 0 (Kohlberg, 1974) and the uniform value does not exists.

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Maxmin and minmax exist in stochastic games with signals.

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An example (Coulomb) is as follows

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a	1^*	0^*	L
b	0	1	L

Payoffs (L large)

	α	β	γ
a	?	?	?
b	A	B	A

Signals

Player 2 will start by playing $(0, \varepsilon, 1-\varepsilon)$ and switch to $(1-\varepsilon, \varepsilon, 0)$ when the probability under σ of *a* is small enough.

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4.4 Conjectures

Conjecture 1:
For all games of the general model with finite sets (parameters, actions, signals)

$$\lim v_n = \lim v_\lambda$$

Conjecture 2:
For all such repeated games where the information of player 1 refines the information of player 2

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5.1 Recursive formula

When dealing with compact case, a recursive structure holds for games described in Section 4 and we follow the proof in Mertens, Sorin and Zamir (1994), Sections III.1, III.2, IV.3.. Consider for example a game with lack of information on one side and with private signals. Given the strategy σ of player 1 and his own signals, player 2 computes posterior probabilities on the state. Since player 1 does not have access to player's 2 signals, these conditional probabilities of player 2 are unknown to player 1, but player 1 has probabilities on them. In addition player 2 has probabilities on those beliefs of player 1 and so on.

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When dealing with compact case, a recursive structure holds for games described in Section 4 and we follow the proof in Mertens, Sorin and Zamir (1994), Sections III.1, III.2, IV.3.. Consider for example a game with lack of information on one side and with private signals. Given the strategy σ of player 1 and his own signals, player 2 computes posterior probabilities on the state. Since player 1 does not have access to player's 2 signals, these conditional probabilities of player 2 are unknown to player 1, but player 1 has probabilities on them. In addition player 2 has probabilities on those beliefs of player 1 and so on.

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The recursive structure thus relies on the construction of the universal belief space, Mertens and Zamir (1985), that represents this infinite hierarchy of beliefs: $\Xi = M \times \Theta^1 \times \Theta^2$, where Θ^i, homeomorphic to $\Delta(M \times \Theta^{-i})$, is the type set of Player i. The signaling structure in the game, just before the moves at stage n, describes an information scheme (basically a probability on $M \times \hat{A} \times \hat{B}$ where $\hat{A}$ is an abstract signal space to player 1 and the same for 2) that induces a consistent distribution on Ξ.

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This is referred to as the entrance law $\mathcal{P}_n \in \Delta(\Xi)$. The entrance law $\mathcal{P}_n$ and the (behavioral) strategies at stage n (say α_n and β_n) from type set to mixed move set determine the current payoff and the new entrance law $\mathcal{P}_{n+1} = H(\mathcal{P}_n, \alpha_n, \beta_n)$. This updating rule is the basis of the recursive structure. The stationary aspect of the repeated game is expressed by the fact that H does not depend on n.

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The Shapley operator is defined on the set of real bounded functions on $\Delta(\Xi)$ by:

$$\Psi(f)(\mathcal{P}) = \sup_{\alpha} \inf_{\beta} \{g(\mathcal{P}, \alpha, \beta) + f(H(\mathcal{P}, \alpha, \beta))\}$$

and the usual relations hold, see Mertens, Sorin and Zamir, (1994) Section IV.3:

$$(n+1)v_{n+1}(\mathcal{P}) = \mathtt{val}_{\alpha\times\beta}\{g(\mathcal{P}, \alpha, \beta) + nv_n(H(\mathcal{P}, \alpha, \beta))\}$$

$$v_\lambda(\mathcal{P}) = \mathtt{val}_{\alpha\times\beta}\{\lambda g(\mathcal{P}, \alpha, \beta) + (1-\lambda)v_\lambda(H(\mathcal{P}, \alpha, \beta))\}.$$

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We have here a "deterministic" stochastic game: in the framework of a regular stochastic game, it would correspond to working at the level of distributions on the state space, $\Delta(K)$.
We still have the ε-weighted operator related to the initial Shapley operator by :

$$\Phi(\varepsilon, f) = \varepsilon \Psi\left(\frac{(1-\varepsilon)f}{\varepsilon}\right). \tag{8}$$

so that

$$v_n = \Phi\left(\frac{1}{n}, v_{n-1}\right), \qquad v_\lambda = \Phi(\lambda, v_\lambda) \tag{9}$$

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The asymptotic study relies thus on the behavior of $\Phi(\varepsilon, \cdot)$, as ε goes to 0. Obviously if v_n or v_λ converges uniformly, the limit w will satisfy :

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A general result in this framework is

Theorem

Neyman (2003)
If v_λ is of bounded variation in the sense that for any sequence λ_i decreasing to 0

$$\sum_i \|v_{\lambda_{i+1}} - v_{\lambda_i}\| < \infty \tag{11}$$

then $\lim_{n\to\infty} v_n = \lim_{\lambda\to 0} v_\lambda$.

The explicit construction of the recursive structure in the framework of repeated games with incomplete information is as follows. A one-stage strategy of Player 1 is an element x in $\mathbf{X} = \Delta(I)^K$ (resp. y in $\mathbf{Y} = \Delta(J)^L$ for Player 2).

We represent now this game as a stochastic game. The basic state space is $\chi = \Delta(K) \times \Delta(L)$ and corresponds to the beliefs of the players on the parameter along the play. The transition is given by a map Π from $\chi \times \mathbf{X} \times \mathbf{Y}$ to probabilities on χ with $\Pi((p(i), q(j)) | (p, q), x, y) = x(i)y(j)$, where $p(i)$ is the conditional probability on K given the move i and $x(i)$ the probability of this move (and similarly for the other variable).

Explicitely: $x(i) = \sum_k p^k x_i^k$ and $p^k(i) = \dfrac{p^k x_i^k}{x(i)}$.

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Ψ is now an operator on the set of real bounded saddle (concave/convex) functions on χ, Rosenberg and Sorin (2001):

$$\Psi(f)(p,q) = \mathtt{val}_{\mathbf{X}\times\mathbf{Y}}\{g(p,q,x,y)+\int_{\chi} f(p',q')\Pi(d(p',q')|(p,q),x,y)\} \tag{12}$$

with $g(p,q,x,y) = \sum_{k,\ell} p^k q^\ell g(k,\ell,x^k,y^\ell)$.
Note that by the definition of Π, the state variable is updated as function of the one-stage strategies of the players, which are not public information during the play, nor are the strategies α and β introduced above.

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The argument is thus first to prove the existence of a value (v_n or v_λ) and then using the minmax theorem to construct an equivalent game, in the sense of having the same sequence of values, in which one-stage strategies are announced. This last game is now reducible to a (deterministic) stochastic game played at the distribution's level.

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5.2 Variational inequalities

We use the operator approach to obtain properties on the asymptotic value, following Rosenberg and Sorin (2001).

a) Existence of the asymptotic value

We first introduce sets of functions that will correspond to upper and lower bounds on the sequences of values. This allows, for certain classes of games, to identify the asymptotic value through variational inequalities.

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Theorem

If f satisfies: for all $\delta > 0$ there exists R_δ such that $R \geq R_\delta$ implies

$$\Psi(Rf) \leq (R+1)f + \delta \tag{13}$$

$\limsup_{n\to\infty} v_n$ *and* $\limsup_{\lambda\to 0} v_\lambda$ *are less than f.*

This allows to obtain $\lim v_n = \lim v_\lambda$ in absorbing and recursive games with finite state space and compact action spaces.

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This allows to obtain $\lim v_n = \lim v_\lambda$ in absorbing and recursive games with finite state space and compact action spaces.

More generally, when the state space is not finite, one can introduce the larger class of functions $\mathcal{S}^+$ where in condition (13) only simple convergence is required:

$$\theta^+(f)(k) = \limsup_{R\to\infty}\{\Psi(Rf)(k) - (R+1)f(k)\} \leq 0 \qquad \forall k \quad (14)$$

In the case of continuous functions on a compact set K, an argument similar to the maximum principle implies that $f^+ \geq f^-$ for any functions $f^+ \in \mathcal{S}^+$ and $f^- \in \mathcal{S}^-$ (defined similarly with $\theta^-(f^-) \leq 0$). Hence the intersection of the closures of $\mathcal{S}^+$ and $\mathcal{S}^-$ contains at most one point.

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This argument suffices for the class of games with incomplete information on both sides: any accumulation point w of the family v_λ as $\lambda \to 0$ belongs to the closure of $\mathcal{S}^+$, hence by symmetry the existence of a limit follows, (Rosenberg and Sorin, 2001).

In the framework of (finite) absorbing games with incomplete information on one side, where the parameter is both changing and unknown, Rosenberg (2000) used similar tools in a very sophisticated way to obtain the first general results of existence of an asymptotic value concerning this class of games.

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b) The derived game: characterization of the asymptotic value

Still dealing with the Shapley operator, one can use the existence of a limit:

$$\varphi(f)(k) = \lim_{\varepsilon \to 0^+} \frac{\Phi(\varepsilon, f)(k) - \Phi(0, f)(k)}{\varepsilon}$$

$\varphi(f)(k)$ being the value of the "derived game" with payoff $h(k,x,y) = g(k,x,y) - E_{k,x,y} f$, played on the product of the subsets of optimal strategies in $\Phi(0, f)$.

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The relation with (14) is given by:

$$\theta^{+}(f) = \theta^{-}(f) = \begin{cases} \varphi(f) & \textit{if } \Phi(0,f) = f \\ +\infty & \textit{if } \Phi(0,f) > f \\ -\infty & \textit{if } \Phi(0,f) < f \end{cases}$$

In the setup of games with incomplete information one shows that any accumulation point of the sequence of values is closed to a function f with $\varphi(f) \leq 0$ and this implies the existence of the asymptotic value.

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Let $\mathcal{E}f$ being the projection on K of the extreme points of the epigraph of f. Then $v = \lim_{n\to\infty} v_n = \lim_{\lambda\to 0} v_\lambda$ is a saddle continuous function satisfying both inequalities:

$$p \in \mathcal{E}v(\cdot, q) \Rightarrow v(p,q) \leq u(p,q)$$

$$q \in \mathcal{E}v(p, \cdot) \Rightarrow v(p,q) \geq u(p,q) \tag{15}$$

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Let $\mathcal{E}f$ being the projection on K of the extreme points of the epigraph of f. Then $v = \lim_{n\to\infty} v_n = \lim_{\lambda\to 0} v_\lambda$ is a saddle continuous function satisfying both inequalities:

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6.1 Incomplete information, convexity and duality

Consider a two person zero sum game with incomplete information on one side defined by sets of actions S and T, a finite parameter space K, a probability p on K and for each k a real payoff function G^k on $S \times T$. Assume S and T convex and for each k, G^k bounded and bilinear on $S \times T$. The game is played as usual: $k \in K$ is selected according to p and told to player 1 (the maximizer) while player 2 only knows p.
In normal form, Player 1 chooses $\mathbf{s} = \{s^k\}$ in S^K, Player 2 chooses t in T and the payoff is

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Let

$$\underline{v}(p) = \sup_{S^K} \inf_{T} G^p(\mathbf{s}, t) \qquad \overline{v}(p) = \inf_{T} \sup_{S^K} G^p(\mathbf{s}, t)$$

then both $\underline{v}$ and $\overline{v}$ are concave.

Following De Meyer (1996a) one introduces, given $z \in \boldsymbol{R}^k$, the "dual game" $G^*(z)$, where player 1 chooses k and plays s in S while player 2 plays t in T and the payoff is

$$h[z](k,s;t) = G^k(s,t) - z^k.$$

Define by $\underline{w}(z)$ and $\overline{w}(z)$ the corresponding maxmin and minmax which are convex.

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Theorem

The following duality relations holds:

$$\hat{w}(z) = \max_{p \in \Delta(K)} \{\hat{v}(p) - \langle p, z \rangle\} \tag{16}$$

$$\hat{v}(p) = \inf_{z \in \mathbf{R}^K} \{\hat{w}(z) + \langle p, z \rangle\} \tag{17}$$

where $\hat{f}$ stands for $\underline{f}$ or $\overline{f}$.

Given z, let p achieve the maximum in (16) and $\mathbf{s}$ be ε-optimal in G^p: then $(p, \mathbf{s})$ is ε-optimal in $G^*(z)$.

Given p, let z achieve the infimum up to ε in (17) and t be ε-optimal in $G^*(z)$: then t is also 2ε-optimal in G^p.

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6.2 The dual of a repeated game with incomplete information

We consider now repeated games with incomplete information on one side.

a) Primal and dual recursive equations

The use of the dual game will be of interest for two purposes: construction of optimal strategies for the uninformed player and asymptotic analysis.

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Theorem

The maxmin satisfies in the primal game:

$$(n+1)\underline{v}_{n+1}(p) = \max_{x \in X^K} \min_{y \in Y} \{\sum_k p^k x^k G^k y + n \sum_i \hat{x}(i) \underline{v}_n(p(i))\} \qquad (18)$$

with $\hat{x}(i) = \sum_k p^k x^k(i)$ *and* $p^k(i)$ *= Prob* $(k|i)$.
The minmax satisfies in the dual game:

$$(n+1)\overline{w}_{n+1}(z) = \min_{y \in Y} \max_{i \in I} n\overline{w}_n(\frac{n+1}{n} z - \frac{1}{n} G_i y). \qquad (19)$$

Each of these ‘dynamic programming type ” equations allows to construct optimal strategies for the player operating first but the state variable is not the same.

The main advantage of dealing with (19) rather than with (18) is that the state variable evolves smoothly from z to $z + \frac{1}{n}(z - G_i y)$.

Rosenberg (1998) extended the previous duality to games having at the same time incomplete information and stochastic transition on the parameters which allows to deduce properties of optimal strategies in this dual game for each player.

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b) The differential dual game

This section follows Laraki (2002) and starts again from equation (19). The recursive formula for the value w_n of the dual of the n stage game can be written, since $w_n(z)$ is convex, as:

$$(n+1)w_{n+1}(z) = \min_{y \in Y} \max_{x \in X} n w_n\left(\frac{n+1}{n} z - \frac{1}{n} xGy\right). \tag{20}$$

This leads to consider w_n as the n^{th} discretization of the upper value of the differential game (of fixed duration) on $[0, 1]$ with dynamic $\zeta(t) \in R^K$ given by:

$$\frac{d\zeta}{dt} = x_t G y_t, \quad \zeta(0) = -z$$

$x_t \in X$, $y_t \in Y$ and terminal payoff $\max_k \zeta^k(1)$.

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Basic results of the theory of differential games (see e.g. Souganidis (1999)) show that the game starting at time t from state ζ has a value $\varphi(t,\zeta)$, which is the only viscosity solution, uniformly continuous in ζ uniformly in t, of the following partial differential equation with boundary condition:

$$\frac{\partial \varphi}{\partial t} + u(\nabla \varphi) = 0, \quad \varphi(1,\zeta) = \max_k \zeta^k. \tag{21}$$

One recovers also the speed of convergence and the identification of the limit trough variational inequalities in terms of local sub- and super-differentials.
Similar tools have been recently introduced by Cardaliaguet (2007, 2008, 2009) to study differential games of fixed duration and incomplete information on both sides $\Gamma(p,q)[\theta,t]$.

7.1 Dynamic programming and MDP

In the framework of general dynamic programming (one person stochastic game with a state space Ω, a correspondence C from Ω to itself and a real bounded payoff g on Ω) Lehrer and Sorin (1992) gave an exemple where $\lim_{n\to\infty} v_n$ and $\lim_{\lambda\to 0} v_\lambda$ both exist and differ.

They also proved that uniform convergence (on Ω) of v_n is equivalent to uniform convergence of v_λ and then the limits are the same.

However this condition alone does not imply existence of the uniform value, v_∞, see Lehrer and Monderer (1994), Monderer and Sorin (1993).

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Recent advances have been obtained by Renault (2007a) introducing new notions like the values v_{nm} (resp. ν_{nm}) of the game where the payoff is the average between stage $n+1$ and $n+m$ (resp. the mininum of all averages between stage $n+1$ and $n+\ell$ for $\ell \leq m$).

Theorem

Assume that the state space Ω is metric compact and the family of functions v_{nm} and ν_{nm} are uniformy equicontinuous. Then the uniform value v_∞ exits.

Player 1 cannot get more than $\min_m \max n \nu_{nm}$ and under the above condition this quantity is also $\max n \min_m \nu_{nm}$ (and the same with ν replace by v).

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In particular when applied to MDP (finite state space K, action space I, signal space A and transition from $K \times I$ to $K \times A$) the previous result implies

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General MDP processes with finite state space have a uniform value

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7.2 Games with transition controlled by one player

Consider now a game where player 1 controls the transition on the state:
basic examples are stochastic games where the transition is independent of player's 2 moves, or games with incomplete information on one side (with no signals)
but this class also covers the case where the state is random, its evolution independent of player 2's moves and player 1 knows more than player 2.
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One reduces the analysis of the game to a dynamic programming problem by looking at stage by stage best reply of player 2 (whose moves do not affect the future of the process) and the finiteness assumption on the basic datas implies

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Renault (2007b)
In the finite case, games with transition controlled by one player have a uniform value.

The result extends previous work of Rosenberg, Solan and Vieille (2004) and Renault (2006), but explicit formulas are not yet available (Marino, 2005, Horner J., D. Rosenberg, E. Solan and N. Vieille, 2006, Neyman, 2005)

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Unification of the field: same tools for the general model including stochastic aspects, incomplete information and signals
Similar approach for games in continuous time or differential games

Compact case: recursive formula
general evaluation function
new tools: viscosity solutions and comparison arguments
link with differential games of fixed duration
example: weak approachability
study of the limit game

Uniform approach
regularity of the family of compact value functions and level of information
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Bibliography I

- Assoulamani S., M. Quincampoix and S. Sorin (2009) Repeated games and qualitative differential games: approachability and comparison of strategies, *SIAM J. on Control and Optimization*, **48**, 2461-2479.
- Aumann R.J. and M. Maschler (1995), *Repeated Games with Incomplete Information*, M.I.T. Press (with the collaboration of R. Stearns).
- Bardi M. and I. Capuzzo Dolcetta (1996) *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Birkhauser.

Bibliography II

- Bewley T. and E. Kohlberg (1976a), The asymptotic theory of stochastic games, *Mathematics of Operations Research*, **1**, 197-208.
- Bewley T. and E. Kohlberg (1976b), The asymptotic solution of a recursion equation occurring in stochastic games, *Mathematics of Operations Research*, **1**, 321-336.
- Blackwell D. (1956), An analog of the minmax theorem for vector payoffs, *Pacific Journal of Mathematics*, **6**, 1-8.
- Blackwell D. and T. Ferguson (1968), The Big Match, *Annals of Mathematical Statistics*, **39**, 159-163.

Bibliography III

- Cardaliaguet P. (1996) A differential game with two players and one target, *SIAM J. on Control and Optimization*, **34**, 1441-1460.
- Cardaliaguet P. (2007) Differential games with asymmetric information, *SIAM J. on Control and Optimization*, **46**, 816-838.
- Cardaliaguet P. (2008) Representation formulas for differential games with asymmetric information, *JOTA*, **138**, 1-16.

Bibliography IV

- Cardaliaguet P. (2009) Numerical approximation and optimal strategies for dfferential games with lack of information on one side, *Advances in Dynamic Games and their Applications*, Bernhard P., V. Gaitsgory and O. Pourtalier (eds.), Annals of ISDG, **10**, Birkhauser, 159-176.
- Cardaliaguet P. and M. Quincampoix (2008) Deterministic differential games under probability knowledge of initial condition, *International Game Theory Review*, **10**, 1-16.

Bibliography V

- Cardaliaguet P. and C. Rainer (2009a) Stochastic differential games with asymmetric information, *Appl. Math. Optim.*, **59**, 1-36.
- Cardaliaguet P. and C. Rainer (2009b) On a continuous time game with incomplete information, *Mathematics of Operations Research*, **34**, 769-794.
- Coulomb J.-M. (1992), Repeated games with absorbing states and no signals, *International Journal of Game Theory*, **21**, 161-174.

Bibliography VI

- Coulomb J.-M. (1996), A note on 'Big Match', *ESAIM: Probability and Statistics*, **1**, 89-93, http://www.edpsciences.com/ps/.
- Coulomb, J.-M. (1999), Generalized Big Match, *Mathematics of Operations Research*, **24**, 795-816.
- Coulomb, J.-M. (2001), Repeated games with absorbing states and signalling structure, *Mathematics of Operations Research*, **26**, 286-303.

Bibliography VII

- Coulomb J.M. (2003) Stochastic games without perfect monitoring, *International Journal of Game Theory*, **32**, 73-96.
- De Meyer B. (1996a), Repeated games and partial differential equations, *Mathematics of Operations Research*, **21**, 209-236.
- De Meyer B. (1996b), Repeated games, duality and the Central Limit theorem, *Mathematics of Operations Research*, **21**, 237-251.

Bibliography VIII

- De Meyer B. (1999), From repeated games to Brownian games, *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, **35**, 1-48.
- De Meyer B. and D. Rosenberg (1999), "Cav u" and the dual game, *Mathematics of Operations Research*, **24**, 619-626.
- Everett H. (1957), Recursive games, in *Contributions to the Theory of Games, III*, M. Dresher, A.W. Tucker and P. Wolfe (eds.), Annals of Mathematical Studies, 39, Princeton University Press, 47-78.

Bibliography IX

- Horner J., D. Rosenberg, E. Solan, N. Vieille (2010) On a Markov game with one-sided incomplete information, *Operations Research*, forthcoming.
- Kohlberg E. (1974), Repeated games with absorbing states, *Annals of Statistics*, **2**, 724-738.
- Laraki R. (2001a), Variational inequalities, systems of functional equations and incomplete information repeated games, *SIAM Journal of Control and Optimization*, **40**, 516-524.

Bibliography X

- Laraki R. (2001b), The splitting game and applications, *International Journal of Game Theory*, **30**, 359-376.
- Laraki R. (2002), Repeated games with lack of information on one side: the dual differential approach, *Mathematics of Operations Research*, **27**, 419-440.
- Laraki R. (2010a) Explicit formulas for repeated games with absorbing states, *International Journal of Game Theory,* **39**, 53-69.

Bibliography XI

- Lehrer E. and D. Monderer (1994), Discounting versus averaging in dynamic programming, *Games and Economic Behavior*, **6**, 97-113.
- Lehrer E. and S. Sorin (1992), A uniform Tauberian theorem in dynamic programming, *Mathematics of Operations Research*, **17**, 303-307.
- Maitra A. and W. Sudderth (1998), Finitely additive stochastic games with Borel measurable payoffs, *International Journal of Game Theory*, **27**, 257-267.

Bibliography XII

- Marino A. (2005) The value and optimal strategies of a particular Markov chain game, preprint.
- Mertens J.-F. (1972), The value of two-person zero-sum repeated games: the extensive case, *International Journal of Game Theory*, **1**, 217-227.
- Mertens J.-F. (1987), Repeated games, in *Proceedings of the International Congress of Mathematicians, Berkeley, 1986*, A. M. Gleason (ed.), American Mathematical Society, 1528-1577.

Bibliography XIII

- Mertens J.-F. (1998), The speed of convergence in repeated games with incomplete information on one side, *International Journal of Game Theory*, **27**, 343-359.
- Mertens J.-F. and A. Neyman (1981), Stochastic games, *International Journal of Game Theory*, **10**, 53-66.
- Mertens J.-F., S. Sorin and S. Zamir (1994), *Repeated Games*, CORE D.P. 9420-21-22.
- Mertens J.-F. and S. Zamir (1971), The value of two-person zero-sum repeated games with lack of information on both sides, *International Journal of Game Theory*, **1**, 39-64.

Bibliography XIV

- Mertens J.-F. and S. Zamir (1976a), The normal distribution and repeated games, *International Journal of Game Theory*, **5**, 187-197.
- Mertens J.-F. and S. Zamir (1976b), On a repeated game without a recursive structure, *International Journal of Game Theory*, **5**, 173-182.
- Mertens J.-F. and S. Zamir (1985), Formulation of Bayesian analysis for games with incomplete information, *International Journal of Game Theory*, **14**, 1-29.

Bibliography XV

- Mills H. D. (1956), Marginal values of matrix games and linear programs, in *Linear Inequalities and Related Systems*, H. W. Kuhn and A. W. Tucker (eds.), Annals of Mathematical Studies, 38, Princeton University Press, 183-193.
- Monderer D. and S. Sorin (1993), Asymptotic properties in dynamic programming, *International Journal of Game Theory*, **22**, 1-11.

Bibliography XVI

- Neyman A. (2003) Stochastic games and nonexpansive maps, *Stochastic Games and Applications*, Neyman A. and S. Sorin (eds.), NATO Science Series C 570, Kluwer Academic Publishers, 397-415.
- Neyman A. (2008), Existence of optimal strategies in Markov games with incomplete information, *International Journal of Game Theory*, **37**, 581-596.
- Neyman A. and S. Sorin (eds.) (2003) *Stochastic Games and Applications*, NATO Science Series C 570, Kluwer Academic Publishers.

Bibliography XVII

- Neyman A. and S. Sorin (2010) Repeated games with public uncertain duration process, *International Journal of Game Theory*, **39**, 29-52.
- Ponssard J.-P. and S. Sorin (1982), Optimal behavioral strategies in zero-sum games with almost perfect information, *Mathematics of Operations Research*, **7**, 14-31.
- Renault J. (2006) The value of Markov chain games with lack of information on one side, *Mathematics of Operations Research*, **31**, 490-512.

Bibliography XVIII

- Renault J. (2007) Uniform value in dynamic programming, revised version: http://arxiv.org/abs/0803.2758..
- Renault J. (2009) The value of repeated games with an informed controller, preprint.
- Rosenberg D. (1998), Duality and Markovian strategies, *International Journal of Game Theory*, **27**, 577-597.
- Rosenberg D. (2000) Zero-sum absorbing games with incomplete information on one side: asymptotic analysis, *SIAM Journal on Control and Optimization*, **39**, 208-225.

Bibliography XIX

Rosenberg D., E. Solan and N. Vieille (2002) Blackwell optimality in Markov decision processes with partial observation, *Annals of Statistics*, **30**, 1178-1193.

Rosenberg D., E. Solan and N. Vieille (2003) The maxmin value of stochastic games with imperfect monitoring, *International Journal of Game Theory*, **32**, 133-150.

Rosenberg D., E. Solan and N. Vieille (2004) Stochastic games with a single controller and incomplete information, *Siam J. Control Opt.*, **43**, 86-110.

Bibliography XX

- Rosenberg D. and S. Sorin (2001), An operator approach to zero-sum repeated games, *Israel Journal of Mathematics*, **121**, 221- 246.
- Rosenberg D. and N. Vieille (2000), The maxmin of recursive games with lack of information on one side, *Mathematics of Operations Research*, **25**, 23-35.
- Shapley L. S. (1953), Stochastic games, *Proceedings of the National Academy of Sciences of the U.S.A*, **39**, 1095-1100.

Bibliography XXI

- Sorin S. (1984), Big Match with lack of information on one side (Part I), *International Journal of Game Theory*, **13**, 201-255.
- Sorin S. (1985), Big Match with lack of information on one side (Part II), *International Journal of Game Theory*, **14**, 173-204.
- Sorin S. (1989), On repeated games without a recursive structure: existence of $\lim v_n$, *International Journal of Game Theory*, **18**, 45-55.

Bibliography XXII

- Sorin S. (2002), *A First Course on Zero-Sum Repeated Games*, Springer.
- Sorin S. (2003), The operator approach to zero-sum stochastic games, Chapter 27 in *Stochastic Games and Applications*, A. Neyman and S. Sorin (eds.), NATO Science Series C 570, Kluwer Academic Publishers.
- Sorin S. (2004), Asymptotic properties of monotonic nonexpansive mappings, *Discrete Events Dynamic Systems*, **14**, 109-122.

Bibliography XXIII

- Sorin S. (2005) New approaches and recent advances in two-person zero-sum repeated games, *Advances in Dynamic Games*, Annals of the International Society of Dynamic Games , 7 Nowak A.S. and K. Szajowski (eds.), Birkhauser, 67-93.
- Sorin S., X. Venel and G. Vigeral (2010) Asymptotic properties of optimal trajectories in dynamic programming, *Sankhya : The Indian Journal of Statistics*, **72A**, 237-245.

Bibliography XXIV

- Sorin S. (2011) Zero-Sum Repeated Games: Recent Advances and New Links with Differential Games, *Dynamic Games and Applications*, **1**, 172-207 ..
- Souganidis P.E. (1985), Approximation schemes for viscosity solutions of Hamilton-Jacobi equations, *Journal of Differential Equations*, **17**, 781-791.
- Souganidis P.E. (1999), Two player zero sum differential games and viscosity solutions, in *Stochastic and Differential Games*, M. Bardi, T.E.S. Raghavan and T. Parthasarathy (eds.), Birkhauser, 70-104.

Bibliography XXV

- Souquière A. (2010) Approximation and representation of the value for some differential games with asymmetric information, *International Journal of Game Theory*, to appear.
- Spinat X. (2002) *A necessary and sufficient condition for approachability*, Mathematics of Operations Research, **27**, 31-44.
- Vieille N. (1992), Weak approachability, *Mathematics of Operations Research*, **17**, 781-791.

Bibliography XXVI

Vigeral G. (2010a) Evolution equations in discrete and continuous time for non expansive operators in Banach spaces, *ESAIM COCV*, to appear.

Waternaux C.(1983), Solution for a class of repeated games without recursive structure, *International Journal of Game Theory*, **12**, 129-160.

Zamir S. (1973), On the notion of value for games with infinitely many stages, *Annals of Statistics*, **1**, 791-796.