Zero-Sum Repeated Games: Basic Results and New Advances

Sylvain Sorin
UPMC-Paris 6 and Ecole Polytechnique
sorin@math.jussieu.fr
http://www.math.jussieu.fr/ sorin/

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Repeated games modelize situations involving multistage interaction, where at each period the players are facing a stage game in which their actions have two effects: they induce a stage payoff and they affects the future of the game. Note the difference with other multimove games like pursuit or stopping games.

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The case is drastically different for non-zero-sum games leading to the whole family of so called “Folk theorem” and its variants: the use of plans and threats allows to generate new equilibria.
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Here we will concentrate on the zero-sum case.
Recall the simplest case where the game is defined by a $I \times J$ real valued matrix. $A_{ij}$ describes the gain of player 1 (and the loss of player 2) if player 1 (resp. 2) chooses the move $i$ (resp. $j$).
Let $\Delta(I) = X$, the simplex on $I$, denote the set of mixed moves (and similarly $\Delta(J) = Y$) and extend $A$ to $X \times Y$ in a bilinear way.
The celebrated minmax theorem (von Neumann, 1928) states that there exists a triple $(v, x, y) \in \mathcal{R} \times X \times Y$ with
\[
xAy' \geq v, \forall y' \in Y, x'Ay \leq v, \forall x' \in X.
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$$x Ay' \geq v, \forall y' \in Y, x' Ay \leq v, \forall x' \in X.$$
More generally one considers a zero-sum game defined by a map $F : X \times Y \rightarrow \mathbb{R}$.

Player 1 can obtain: $v = \sup_X \inf_Y F(x, y)$ and Player 2: $\bar{v} = \inf_Y \sup_X F(x, y)$.

The game has a value when $v = \bar{v}$.

A general result is due to Sion (1958) and involves geometric conditions:

$X$ and $Y$ convex, $F$ quasiconcave in $x$ and quasiconvex in $y$ and topological conditions $X$ or $Y$ compact, $F$ upper semicontinuous in $x$ and lower semicontinuous in $y$. 
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$X$ and $Y$ convex, $F$ quasiconcave in $x$ and quasiconvex in $y$ and topological conditions $X$ or $Y$ compact, $F$ upper semicontinuous in $x$ and lower semicontinuous in $y$. 
We now consider the case where the stage game belongs to a family $G^k$, $k \in K$ of two-person zero-sum games played on strategy sets $I \times J$. The two basic classes of repeated games that have been extensively studied are stochastic games and incomplete information games. In the first class, the parameter $k$ is a publicly known variable, controlled by the players. It evolves along a play and its value at stage $n+1$, called the state $k_{n+1}$, is stationary random function of the triple $(i_n, j_n, k_n)$ which are the moves, respectively the state, at stage $n$. At each stage both players share the same information and in particular they know the current state.
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However this one is changing and the issue for the players at stage $n$ is to control both the current payoff $g_n$ (induced by $(i_n, j_n, k_n)$) and the next state $k_{n+1}$.

In the second class, the parameter $k$ is chosen once for all and kept fixed during the play. However, at least one player does not have full information about it. In this framework, the issue is the tradeoff between using the information (this increases the set of strategies which are the probability distribution of the moves, given the past) and revealing it.

We will see later on that these two models apparently very different - state known and changing versus state unknown and fixed- are in fact particular incarnations of a much more general model and that they share a lot of properties.
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A play of the game generates a sequence of stagepayoffs \( \{g_n\} \). There are several ways of comparing outcomes in such a framework.

We first introduce the **compact case**. For every probability distribution \( \mu \) on the integers \( n \geq 1 \) (\( \mu(n) \geq 0, \sum_n \mu(n) = 1 \)), one can define a game \( \Gamma[\mu] \) with evaluation \( \sum_n g_n \mu(n) \).

Under standard assumptions on the basic data, the natural topology on plays induces a game with compact strategy spaces and continuous payoff function, hence the value \( v[\mu] \) will exist by Sion’s theorem.

The **asymptotic approach** studies the family of such games as the expected length (the mean of \( \mu \)) goes to \( +\infty \).
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Two typical examples correspond to:
1) the **finite \( n \)-stage** game \( \Gamma_n \) with outcome given by the average of the first \( n \) payoffs:

\[
\gamma_n = \frac{1}{n} \sum_{t=1}^{n} g_t
\]

2) the **\( \lambda \)-discounted game** \( \Gamma_\lambda \) with outcome equal to the discounted sum of the payoffs:

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\gamma_\lambda = \sum_{t=1}^{\infty} \lambda(1 - \lambda)^{t-1} g_t
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The values of these games are denoted by \( v_n \) and \( v_\lambda \) respectively. The study of their asymptotic behavior, as \( n \) goes to \( \infty \) or \( \lambda \) goes to 0 is one example of the study of the asymptotic game.
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The values of these games are denoted by $v_n$ and $v_\lambda$ respectively. The study of their asymptotic behavior, as $n$ goes to $\infty$ or $\lambda$ goes to 0 is one example of the study of the asymptotic game.
Extensions consider games with random duration process where the weight $\mu(n)$ is a random variable which law depends upon the previous path on the random duration tree (Neyman, 2003, Neyman and Sorin, 2010). Note that the knowledge of the duration (i.e. the evaluation process) is crucial in the definition of the strategies.
An alternative analysis, called the **uniform approach**, considers the whole family of “long games”.

It does not specify outcome in some infinitely repeated game like $\lim \inf \frac{1}{n} \sum_{t=1}^{n} g_t$ or a measurable function defined on plays (see Maitra and Sudderth, 1998), but look for strategies exhibiting asymptotic uniform properties in the following sense: they are optimal in any sufficiently long game.
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Explicitly, \( \underline{v} \) is the \textbf{maxmin} if the two following conditions are satisfied:

- Player 1 can \textbf{guarantee} \( \underline{v} \): for any \( \varepsilon > 0 \), there exists a strategy \( \sigma \) of Player 1 and an integer \( N \) such that for any \( n \geq N \) and any strategy \( \tau \) of Player 2:

\[
\gamma_n(\sigma, \tau) \geq \underline{v} - \varepsilon
\]

where \( \gamma_n(\sigma, \tau) \) is the expectation of \( \gamma_n \) under the pair of strategies \((\sigma, \tau)\).

- Player 2 can \textbf{defend} \( \underline{v} \): for any \( \varepsilon > 0 \) and any strategy \( \sigma \) of Player 1, there exist an integer \( N \) and a strategy \( \tau \) of Player 2 such that for all \( n \geq N \):

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\gamma_n(\sigma, \tau) \leq \underline{v} + \varepsilon.
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Explicitely, $\overline{v}$ is the **maxmin** if the two following conditions are satisfied:

- Player 1 can **guarantee** $\overline{v}$: for any $\varepsilon > 0$, there exists a strategy $\sigma$ of Player 1 and an integer $N$ such that for any $n \geq N$ and any strategy $\tau$ of Player 2:

$$\gamma_n(\sigma, \tau) \geq \overline{v} - \varepsilon$$

where $\gamma_n(\sigma, \tau)$ is the expectation of $\gamma_n$ under the pair of strategies $(\sigma, \tau)$.

- Player 2 can **defend** $\overline{v}$: for any $\varepsilon > 0$ and any strategy $\sigma$ of Player 1, there exist an integer $N$ and a strategy $\tau$ of Player 2 such that for all $n \geq N$:

$$\gamma_n(\sigma, \tau) \leq \overline{v} + \varepsilon.$$
Explicitely, $\nu$ is the **maxmin** if the two following conditions are satisfied:
- Player 1 can **guarantee** $\nu$: for any $\varepsilon > 0$, there exists a strategy $\sigma$ of Player 1 and an integer $N$ such that for any $n \geq N$ and any strategy $\tau$ of Player 2:
  $$\gamma_n(\sigma, \tau) \geq \nu - \varepsilon$$
  where $\gamma_n(\sigma, \tau)$ is the expectation of $\gamma_n$ under the pair of strategies $(\sigma, \tau)$.
- Player 2 can **defend** $\nu$: for any $\varepsilon > 0$ and any strategy $\sigma$ of Player 1, there exist an integer $N$ and a strategy $\tau$ of Player 2 such that for all $n \geq N$:
  $$\gamma_n(\sigma, \tau) \leq \nu + \varepsilon.$$
A dual definition holds for the **minmax** $\bar{v}$.

Whenever $v = \bar{v}$, the game has a **uniform value**, denoted by $v_\infty$.

Remark that the existence of $v_\infty$ implies:

$$v_\infty = \lim_{n \to \infty} v_n = \lim_{\lambda \to 0} v_\lambda.$$
A dual definition holds for the minmax $\overline{v}$. Whenever $v = \overline{v}$, the game has a uniform value, denoted by $v_\infty$. Remark that the existence of $v_\infty$ implies:

$$v_\infty = \lim_{n \to \infty} v_n = \lim_{\lambda \to 0} v_\lambda.$$
A dual definition holds for the \textbf{minmax} $\overline{v}$. Whenever $v = \overline{v}$, the game has a \textbf{uniform value}, denoted by $v_\infty$.

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3.1 Stochastic games and Shapley operator

a) Consider a stochastic game with state space $K$, action spaces $I$ and $J$ (all finite) and real payoff function $g$ defined on $K \times I \times J$. The initial state $k_1$ is announced to both players. In addition at each stage $t + 1$, the transition probability $Q(\cdot|k_t, i_t, j_t)$ defines the law of the next state $k_{t+1}$. Assume that the past history $(k_1, i_1, j_1, \cdots, k_t, i_t, j_t, k_{t+1})$ is known by the players at each stage $t + 1$. Introduce the set of mixed (random) moves $X = \Delta(I)$ and $Y = \Delta(J)$ and extend by bilinearity $g$ and $Q$ to $X \times Y$. 
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b) The Shapley operator $\Psi$ acts on the set $\mathcal{F}$ of real functions $f$ on $K$ as follows. First define an auxiliary one shot game on $X \times Y$ by the payoff

$$\Psi_{xy}(f)(k) = g(k, x, y) + \int_{K} f(k')Q(k' | k, x, y) \quad (1)$$

and then let

$$\Psi(f)(k) = \text{val}_{X \times Y} \Psi_{xy}(f)(k) \quad (2)$$

where $\text{val}_{X \times Y}$ stands for the value operator:

$$\text{val}_{X \times Y} = \max_X \min_Y = \min_Y \max_X .$$
b) The Shapley operator $\Psi$ acts on the set $\mathcal{F}$ of real functions $f$ on $K$ as follows. First define an auxiliary one shot game on $X \times Y$ by the payoff

$$\Psi_{xy}(f)(k) = g(k, x, y) + \int_K f(k') Q(k'|k, x, y)$$  \hspace{1cm} (1)$$

and then let

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Shapley (1953)

The discounted stochastic game $\Gamma_\lambda$ has a value which is the unique solution of

$$v_\lambda = \Phi(\lambda, v_\lambda)$$

Both players have stationary optimal strategies: at stage $n$, play optimally in the one shot auxiliary game $\Phi_{xy}(\lambda, v_\lambda)(k_n)$
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Both players have stationary optimal strategies: at stage $n$, play optimally in the one shot auxiliary game $\Phi_{xy}(\lambda, v_\lambda)(k_n)$
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$$\|\Psi(f) - \Psi(g)\| \leq \|f - g\| = \sup_{k \in K} |f(k) - g(k)|.$$ 

hence $\Phi(\varepsilon, .)$ is a contraction and this defines uniquely the fixed point, say $w$. By using the proposed strategy player 1 obtains

$$E[\lambda g_n + (1 - \lambda)w(k_{n+1})] \geq w(k_n)$$

which gives

$$E[\sum_{n} \lambda(1 - \lambda)^{n-1} g_n] \geq w(k_1)$$

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Ψ determines the family of values through:

\[ n \nu_n = \Psi^n(0), \quad \frac{\nu_\lambda}{\lambda} = \Psi((1 - \lambda) \frac{\nu_\lambda}{\lambda}). \]  

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c) The next crucial advance concerns the famous “Big Match” described by the following matrix:

\[
\begin{array}{cc}
\alpha & \beta \\
1^* & 0^* \\
0 & 1 \\
\end{array}
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This corresponds to a stochastic game where, as soon as Player 1 plays \(a\), the game reaches an absorbing state with a constant payoff corresponding to the entry played at that stage. Both the \(n\)-stage value \(v_n\) and the \(\lambda\)-discounted value \(v_\lambda\) are equal to 1/2 and are also independent of the additional information transmitted along the play to the players.
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However the corresponding optimal strategies (play \(a\) with probability \(1/(n+1)\) or \(\lambda/(1+\lambda)\)) do not exhibit good asymptotic properties.

**Theorem**

*Blackwell and Ferguson (1968)*

\(v_\infty\) exists.

The proof relies on the construction of a \(\varepsilon\)-optimal strategy of player 1. Define \(L_n = n(\gamma_n - 1/2)\) and play \(a\) at stage \(n + 1\) with probability \(\max\{M, L_n\}^2\) where \(M\) is a large parameter adjusted to \(\varepsilon\).

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e) The first result concerning the asymptotic approach for general stochastic games is the following:

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\lim v_\lambda \text{ and } \lim v_n \text{ exist and are equal.}
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The proof relies on the fact that the equation \( v_\lambda = \Phi(\lambda, v_\lambda) \) can be written as a finite set of polynomial equalities and inequalities involving \( \{x^k_\lambda, y^k_\lambda, v_\lambda(k), \lambda\} \) thus defines a semi-algebraic set in some \( \mathbb{R}^N \) hence by projection, \( v_\lambda \) has an expansion in power series. The result for \( v_n \) follows from a comparison of the recursive operators.
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f) The most important result in the field is the next general existence result

**Theorem**

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The uniform value \( v_\infty \) exists.

The proof uses the bounded variation property of the family \( v_\lambda \) to define an evaluation function \( \bar{\lambda} \) that will adjust the “horizon” to the current performance. At stage \( n+1 \) compute some statistics \( m_{n+1} \) from the past history of payoffs and states (roughly equal to \( m_n + g_n - v_{\lambda_n}(k_n) \)) and play (once) optimally in the auxiliary game \( \Phi_{xy}(\lambda_{n+1}, v_{\lambda_{n+1}}(k_{n+1})) \) where the discount factor is \( \lambda_{n+1} = \bar{\lambda}(m_{n+1}) \).
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3.2 Games with incomplete information

We consider here the simplest framework: independent case and standard signalling.

The repeated game $\Gamma(p, q)$ is defined by a family $\{ G^{k\ell} \}$, $k \in K$, $\ell \in L$, of two-person zero-sum games played on $I \times J$. All the sets are finite.

In addition two probabilities, $p$ on $K$ and $q$ on $L$ are given, according to which a couple $(k, \ell)$ is selected and player 1 (resp. 2) is informed upon $k$ (resp. $\ell$). Then player 1 (resp. 2) chooses, at each stage $n$, a move $i_n \in I$ (resp. $j_n \in J$) and the only information transmitted to both is the couple $(i_n, j_n)$. The stage payoff, $G_{i_nj_n}^{k\ell}$, is usually unknown.
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Denote by $D(p, q)$ the average game $\sum_{k, \ell} p^k q^\ell G_{k\ell}$ and by $u(p, q)$ its value. Note that this corresponds to the game where none of the players is informed - or is using his information.
a) Lack of information on one side
This is the case where $\#L = 1$ hence we drop this index. Player 1 knows the true game while player 2 only knows the initial prior $p$.

**Theorem**

Aumann and Maschler (1966)

$\lim v_n$ exists and equals $\text{Cav } u$.

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Here the operator $C_{av} f$ stands for the smallest function concave and above $f$ on the simplex $\Delta(K)$. 
The proof relies on two basic arguments in the theory of incomplete information games:

First the informed player can generate any martingale with respect to his private information (splitting lemma); since player 1 can obtain $u$ by not using his information, he can also obtain $C_v u$.

The dual inequality is obtained as follows: knowing the strategy $\sigma$ of player 1, player 2 can compute stage after stage a new posterior probability on $K$. This defines a martingale $p_n$ and its variation at each stage is proportional to the gain that player 1 can achieve by using his information. The total $L^1$ variation of a bounded martingale on $n$ stages being at most of the order of $\sqrt{n}$ this gives a speed of convergence of $1/\sqrt{n}$. 

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Theorem

_Aumann and Maschler (1968)_

*The uniform value* \( v_\infty \) *exists.*

Note that the previous concavification argument for player 1 was not using the duration of the game.

For player 2 a concatenation of optimal strategies in longer and longer games \( \Gamma_n(p) \) allows to guarantee \( C_{av} u \).

An alternative proof is based on an explicit optimal strategy of player 2: one considers the game with vector payoffs that player 2 is facing (namely \( \{ G^k_{inj} \}, k \in K \), at stage \( n \)) and on uses Blackwell’s approachability theorem to show that if \( \langle \alpha, p \rangle \geq u(p) \) on \( \Delta(K) \), then the orthant \( \alpha + R^K \) is approachable by player 2.
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b) Lack of information on both sides
Consider first the uniform approach.

**Theorem**

Aumann, Maschler and Stearns (1967)

\( \bar{v} \) and \( \bar{\nu} \) exist and satisfy

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\bar{v}(p, q) = \text{Cav}_p \text{Vex}_q u(p, q)
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The fact that the maxmin is at least $\text{CavVex } u$ relies on the usual concavification argument: playing as a uninformed player (w.r.t. $\ell$) and not using his private information, player 1 can obtain $\text{Vex } u$.

The fact that player 2 can defend this amount relies on a strategy $\tau$ exhausting from the strategy $\sigma$ of player 1 a maximal amount of information. From some stage on, player 1 is not revealing any information anymore and we are back to a situation with incomplete information on one side.
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Note the 2 points:
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In fact the asymptotic analysis leads to the following result:

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$$\lim v_\lambda \text{ and } \lim v_n \text{ exist, are equal and are the unique solution of the set of functional equations}$$

$$v = \text{Cav}_p \min\{u, v\} \quad v = \text{Vex}_q \min\{u, v\}$$

The basic step is the following result:

If player 1 can obtain asymptotically $w$, he can also obtain asymptotically $\text{Vex}_q \max\{u, w\}$: follow at each stage $n$ an optimal strategy for $u(p_n, q_n)$ as long as this quantity is more than $w(p_n, q_n)$ and switch to an optimal for $w$ otherwise.
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Preliminaries
Evaluation of the payoffs: asymptotic and uniform approaches
Basic results
General model and further results
Recursive formula and operator approach
Incomplete information and dual game
MDP and games with one controller
Recent advances
4.1 General model
Let $M$ be a parameter space and $g$ be a function $g$ from $I \times J \times M$ to $\mathbb{R}$: for each $m \in M$. This defines a two person zero-sum game with action spaces $I$ and $J$ for Player 1 and 2 respectively and payoff function $g$.

(Again to simplify the presentation we will consider the case where all sets are finite, avoiding in particular measurability issues).

The initial parameter $m_1$ is chosen at random and the players receive some initial information about it, say $a_1$ (resp. $b_1$) for player 1 (resp. player 2). This choice is performed according to some probability $\pi$ on $M \times A \times B$, where $A$ and $B$ are the signal sets of each player.
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Then at each stage \( n \), player 1 (resp. 2) chooses a move \( i_n \in I \) (resp. \( j_n \in J \)). In addition, after each stage the players obtain some further information about the previous choice of actions and both the previous and the current values of the parameter. This is represented by a map \( Q \) from \( M \times I \times J \) to probabilities on \( M \times A \times B \). At stage \( n \) given the state \( m_n \) and the moves \((i_n, j_n)\), a triple \((m_{n+1}, a_{n+1}, b_{n+1})\) is chosen at random according to the distribution \( Q(m_n, i_n, j_n) \). The new parameter is \( m_{n+1} \), and the signal \( a_{n+1} \) (resp. \( b_{n+1} \)) is transmitted to player 1 (resp. player 2).
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A play of the game is thus a sequence 
\( m_1, a_1, b_1, i_1, j_1, m_2, a_2, b_2, i_2, j_2, \ldots \) while the information of Player 1 before his play at stage \( n \) is a 1-private history of the form \((a_1, i_1, a_2, i_2, \ldots, a_n)\) and similarly for Player 2.

The corresponding sequence of payoffs is \( g_1, g_2, \ldots \) with 
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A strategy $\sigma$ for player 1 is a map from 1-private histories to $\Delta(I)$, the space of probabilities on the set $I$ of actions: it defines the probability distribution of the stage move as a function of the past known to player 1; $\tau$ is defined similarly for player 2. Such a couple $(\sigma, \tau)$ induces, together with the components of the game, $\pi$ and $Q$, a probability distribution on plays, hence on the sequence of payoffs.
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4.2 Games with incomplete information

The standard model presented above corresponds to the case $m_1 = (k, \ell)$, $\pi = p \times q$, initial information $a_1 = k$, $b_1 = \ell$, then $m_n = m_1$ is constant and the signals reveals the moves $(a_{n+1} = b_{n+1} = (i_n, j_n))$.

We discuss here extensions where the signalling structure is more general.

a) Signals: one side

The basic remark (Aumann and Maschler, 1968) is that the 2 notions of using the information (i.e. the distribution of the move $i$ is a fonction $x^k \in X$ of the parameter) and revealing it are no longer equivalent.
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From the previous analysis relying on martingale arguments it follows that the right notion is "revelation".

Define the set $NR$ of non revealing strategies as the vectors \( \{ x^k \} \) of mixed moves such that all components induce the same signals to player 2 whatever being his move.

\( D(p) \) is the one-shot game where player 1 plays in $NR$ and player 2 plays in $Y$ and $u$ is its value. Then one still has

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b) Signals: both sides

When the signalling structure is independent of the state - meaning that the transmission of information is only through the signals induced by the moves - an analysis similar to the one done in the case of lack of information on one side is available. Player 1 has a set $NR_1$ of non-revealing strategies (that depends only upon the signals to player 2) and similarly for player 2. Then one defines the non-revealing game $D(p, q)$ played on $NR_1 \times NR_2$ with payoff

$$D_{x,y}(p, q) = \sum_{k, \ell} p^k q^{\ell} x^k G^{k\ell} y^{\ell}$$

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The relation with stochastic games can be seen on the following signalling matrices corresponding to a $2 \times 2$ game with $2 \times 2$ states (Sorin, 1989)

\[
\begin{array}{cc|cc}
T & L & T & R \\
P & Q & P & Q \\
\hline
B & L & B & R \\
P & Q & P & Q \\
\end{array}
\]  

$H^{11}$  

$H^{12}$  

$H^{21}$  

$H^{22}$
The analysis of such games leads to the study of the whole class of stochastic games with incomplete information on one side. Only partial results are still available (Sorin, 1984, 1985a, Rosenberg and Vieille, 2000) but important differences appear with the previous sub-classes: the uniform value may not exist and $\lim v_n$ may be a transcendental function of the parameters.
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4.3 Stochastic games

The previous model of stochastic games corresponds to the case where the signals to each player are initially the state \((a_1 = b_1 = m_1)\) and are after each stage \(n\) the previous moves and the current state \((a_{n+1} = b_{n+1} = (i_n, j_n, m_{n+1}))\).
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a) Symmetric case
Consider the case where the state may not be known by the players but their information is symmetric (hence include their moves). The players may collect information upon the current state through their moves and the natural state space is their probability on $M$.

**Theorem**

_in the symmetric case the repeated game $\Gamma$ has a value._


Note that the state space ($\Delta(M)$) is no longer finite but the process is very regular (martingale).
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**Theorem**

In the symmetric case the repeated game $\Gamma$ has a value.


Note that the state space ($\Delta(M)$) is no longer finite but the process is very regular (martingale).
b) Stochastic games with signals

We assume here that the signal to each player reveals the current stage but not necessarily the previous move of the opponent. By the recursive formula for $v_\lambda$ and $v_n$ these quantities are the same; however for example in the Big Match, when Player 1 has no information on Player 2’s moves the max min is 0 (Kohlberg, 1974) and the uniform value does not exists.

It follows that the existence of a uniform value for stochastic games depends on the signalling structure on moves.
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Maxmin and minmax exist in stochastic games with signals.

Theorem

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An example (Coulomb) is as follows:

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<tr>
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<td>a</td>
<td>1*</td>
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Payoffs ($L$ large)

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Player 2 will start by playing $(0, \varepsilon, 1-\varepsilon)$ and switch to $(1-\varepsilon, \varepsilon, 0)$ when the probability under $\sigma$ of $a$ is small enough.
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Payoffs ($L$ large)

Player 2

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Signals

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4.4 Conjectures

Conjecture 1:
For all games of the general model with finite sets (parameters, actions, signals)

\[ \lim v_n = \lim v_{\lambda} \]

Conjecture 2:
For all such repeated games where the information of player 1 refines the information of player 2

\[ v = \lim v_n = \lim v_{\lambda} \]
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5.1 Recursive formula

When dealing with compact case, a recursive structure holds for games described in Section 4 and we follow the proof in Mertens, Sorin and Zamir (1994), Sections III.1, III.2, IV.3..

Consider for example a game with lack of information on one side and with private signals. Given the strategy \( \sigma \) of player 1 and his own signals, player 2 computes posterior probabilities on the state. Since player 1 does not have access to player’s 2 signals, these conditional probabilities of player 2 are unknown to player 1, but player 1 has probabilities on them. In addition player 2 has probabilities on those beliefs of player 1 and so on.
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The recursive structure thus relies on the construction of the universal belief space, Mertens and Zamir (1985), that represents this infinite hierarchy of beliefs: \( \Xi = M \times \Theta^1 \times \Theta^2 \), where \( \Theta^i \), homeomorphic to \( \Delta(M \times \Theta^{-i}) \), is the type set of Player \( i \). The signaling structure in the game, just before the moves at stage \( n \), describes an information scheme (basically a probability on \( M \times \hat{A} \times \hat{B} \) where \( \hat{A} \) is an abstract signal space to player 1 and the same for 2) that induces a consistent distribution on \( \Xi \).
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This is referred to as the entrance law $\mathcal{P}_n \in \Delta(\Xi)$. The entrance law $\mathcal{P}_n$ and the (behavioral) strategies at stage $n$ (say $\alpha_n$ and $\beta_n$) from type set to mixed move set determine the current payoff and the new entrance law $\mathcal{P}_{n+1} = H(\mathcal{P}_n, \alpha_n, \beta_n)$. This updating rule is the basis of the recursive structure. The stationary aspect of the repeated game is expressed by the fact that $H$ does not depend on $n$. 
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The Shapley operator is defined on the set of real bounded functions on $\Delta(\Xi)$ by:

$$\Psi(f)(\mathcal{P}) = \sup_{\alpha} \inf_{\beta} \{ g(\mathcal{P}, \alpha, \beta) + f(H(\mathcal{P}, \alpha, \beta)) \}$$

and the usual relations hold, see Mertens, Sorin and Zamir, (1994) Section IV.3:

$$(n+1)v_{n+1}(\mathcal{P}) = \text{val}_{\alpha \times \beta} \{ g(\mathcal{P}, \alpha, \beta) + nv_n(H(\mathcal{P}, \alpha, \beta)) \}$$

$$v_\lambda(\mathcal{P}) = \text{val}_{\alpha \times \beta} \{ \lambda g(\mathcal{P}, \alpha, \beta) + (1 - \lambda)v_\lambda(H(\mathcal{P}, \alpha, \beta)) \}.$$
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where $\text{val}_{\alpha \times \beta} = \sup_\alpha \inf_\beta = \inf_\beta \sup_\alpha$ is the value operator for the “one stage game on $\mathcal{P}$”.
We have here a “deterministic” stochastic game: in the framework of a regular stochastic game, it would correspond to working at the level of distributions on the state space, Δ(K). We still have the $\varepsilon$-weighted operator related to the initial Shapley operator by:

$$\Phi(\varepsilon, f) = \varepsilon \Psi\left(\frac{1 - \varepsilon}{\varepsilon} f\right).$$

so that

$$v_n = \Phi\left(\frac{1}{n}, v_{n-1}\right), \quad v_\lambda = \Phi(\lambda, v_\lambda)$$

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$$
\Phi(\varepsilon, f) = \varepsilon \Psi\left(\frac{(1 - \varepsilon)f}{\varepsilon}\right).
$$

(8)

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The asymptotic study relies thus on the behavior of $\Phi(\varepsilon, \cdot)$, as $\varepsilon$ goes to 0. Obviously if $v_n$ or $v_\lambda$ converges uniformly, the limit $w$ will satisfy:

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$$w = \Phi(0, w) \quad (10)$$
A general result in this framework is

**Theorem**

*Neyman (2003)*

If $v_\lambda$ is of bounded variation in the sense that for any sequence $\lambda_i$ decreasing to 0

$$\sum_i \|v_{\lambda_{i+1}} - v_{\lambda_i}\| < \infty \quad (11)$$

then $\lim_{n \to \infty} v_n = \lim_{\lambda \to 0} v_\lambda$. 
The explicit construction of the recursive structure in the framework of repeated games with incomplete information is as follows. A one-stage strategy of Player 1 is an element $x$ in $X = \Delta(I)^K$ (resp. $y$ in $Y = \Delta(J)^L$ for Player 2).

We represent now this game as a stochastic game. The basic state space is $\chi = \Delta(K) \times \Delta(L)$ and corresponds to the beliefs of the players on the parameter along the play. The transition is given by a map $\Pi$ from $\chi \times X \times Y$ to probabilities on $\chi$ with $\Pi((p(i), q(j)) | (p, q), x, y) = x(i)y(j)$, where $p(i)$ is the conditional probability on $K$ given the move $i$ and $x(i)$ the probability of this move (and similarly for the other variable).

Explicitely: $x(i) = \sum_k p^k x_i^k$ and $p^k(i) = \frac{p^k x_i^k}{x(i)}$. 
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Explicitely: $x(i) = \sum_k p^k x_i^k$ and $p^k(i) = \frac{p^k x_i^k}{x(i)}$. 
Ψ is now an operator on the set of real bounded saddle (concave/convex) functions on $\chi$, Rosenberg and Sorin (2001):

$$
\Psi(f)(p, q) = \text{val}_{\chi \times \chi} \{ g(p, q, x, y) + \int_{\chi} f(p', q') \Pi (d(p', q') | (p, q, x, y)) \}
$$

(12)

with $g(p, q, x, y) = \sum_{k, \ell} p^k q^\ell g(k, \ell, x^k, y^\ell)$.

Note that by the definition of $\Pi$, the state variable is updated as function of the one-stage strategies of the players, which are not public information during the play, nor are the strategies $\alpha$ and $\beta$ introduced above.
Ψ is now an operator on the set of real bounded saddle (concave/convex) functions on χ, Rosenberg and Sorin (2001):

$$
Ψ(f)(p, q) = \text{val}_{X \times Y}\{g(p, q, x, y) + \int_{\chi} f(p', q')\Pi(d(p', q')|(p, q), x, y)\}
$$

(12)

with $g(p, q, x, y) = \sum_{k, \ell} p^k q^\ell g(k, \ell, x^k, y^\ell)$. Note that by the definition of Π, the state variable is updated as function of the one-stage strategies of the players, which are not public information during the play, nor are the strategies α and β introduced above.
ψ is now an operator on the set of real bounded saddle (concave/convex) functions on χ, Rosenberg and Sorin (2001):

\[
\psi(f)(p, q) = \max_{X \times Y} \{g(p, q, x, y) + \int_X f(p', q')\Pi(d(p', q')|(p, q), x, y)\}
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Note that by the definition of \(\Pi\), the state variable is updated as function of the one-stage strategies of the players, which are not public information during the play, nor are the strategies \(\alpha\) and \(\beta\) introduced above.
The argument is thus first to prove the existence of a value ($v_n$ or $v_\lambda$) and then using the minmax theorem to construct an equivalent game, in the sense of having the same sequence of values, in which one-stage strategies are announced. This last game is now reducible to a (deterministic) stochastic game played at the distribution’s level.
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5.2 Variational inequalities

We use the operator approach to obtain properties on the asymptotic value, following Rosenberg and Sorin (2001).

a) Existence of the asymptotic value

We first introduce sets of functions that will correspond to upper and lower bounds on the sequences of values. This allows, for certain classes of games, to identify the asymptotic value through variational inequalities.
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Theorem

If $f$ satisfies: for all $\delta > 0$ there exists $R_\delta$ such that $R \geq R_\delta$ implies

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\psi(Rf) \leq (R + 1)f + \delta
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\limsup_{n \to \infty} v_n \text{ and } \limsup_{\lambda \to 0} v_\lambda \text{ are less than } f.
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This allows to obtain $\lim v_n = \lim v_\lambda$ in absorbing and recursive games with finite state space and compact action spaces.
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This allows to obtain \( \lim v_n = \lim v_\lambda \) in absorbing and recursive games with finite state space and compact action spaces.
More generally, when the state space is not finite, one can introduce the larger class of functions $S^+$ where in condition (13) only simple convergence is required:

$$
\theta^+(f)(k) = \limsup_{R \to \infty} \{ \Psi(Rf)(k) - (R + 1)f(k) \} \leq 0 \quad \forall k \quad (14)
$$

In the case of continuous functions on a compact set $K$, an argument similar to the maximum principle implies that $f^+ \geq f^-$ for any functions $f^+ \in S^+$ and $f^- \in S^-$ (defined similarly with $\theta^-(f^-) \leq 0$). Hence the intersection of the closures of $S^+$ and $S^-$ contains at most one point.
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This argument suffices for the class of games with incomplete information on both sides: any accumulation point $w$ of the family $v_\lambda$ as $\lambda \to 0$ belongs to the closure of $S^+$, hence by symmetry the existence of a limit follows, (Rosenberg and Sorin, 2001).

In the framework of (finite) absorbing games with incomplete information on one side, where the parameter is both changing and unknown, Rosenberg (2000) used similar tools in a very sophisticated way to obtain the first general results of existence of an asymptotic value concerning this class of games.
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b) The derived game: characterization of the asymptotic value

Still dealing with the Shapley operator, one can use the existence of a limit:

\[ \varphi(f)(k) = \lim_{\varepsilon \to 0^+} \frac{\Phi(\varepsilon, f)(k) - \Phi(0, f)(k)}{\varepsilon} \]

\( \varphi(f)(k) \) being the value of the “derived game” with payoff

\[ h(k, x, y) = g(k, x, y) - E_{k,x,y} f, \]

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The relation with (14) is given by:

\[ \theta^+(f) = \theta^-(f) = \begin{cases} 
\varphi(f) & \text{if } \Phi(0,f) = f \\
+\infty & \text{if } \Phi(0,f) > f \\
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In the setup of games with incomplete information one shows that any accumulation point of the sequence of values is closed to a function \( f \) with \( \varphi(f) \leq 0 \) and this implies the existence of the asymptotic value.
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In the setup of games with incomplete information one shows that any accumulation point of the sequence of values is closed to a function \( f \) with \( \varphi(f) \leq 0 \) and this implies the existence of the asymptotic value.
Let $\mathcal{E}f$ being the projection on $K$ of the extreme points of the epigraph of $f$. Then $v = \lim_{n \to \infty} v_n = \lim_{\lambda \to 0} v_\lambda$ is a saddle continuous function satisfying both inequalities:

\begin{align*}
    p \in \mathcal{E}v(\cdot, q) \Rightarrow v(p, q) &\leq u(p, q) \\
    q \in \mathcal{E}v(p, \cdot) \Rightarrow v(p, q) &\geq u(p, q)
\end{align*}

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There is only such one, Laraki (2001a) and one recovers the famous characterization of $v$ due to Mertens and Zamir (1971).
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The operator approach allows for alternative proof of existence and characterization of the asymptotic value Laraki (2001a) through comparison procedures and can be extended to more general games like splitting games, Laraki (2001b).

Conjecture 3:

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1. Preliminaries
2. Evaluation of the payoffs: asymptotic and uniform approaches
3. Basic results
4. General model and further results
5. Recursive formula and operator approach
6. Incomplete information and dual game
7. MDP and games with one controller
8. Recent advances
6.1 Incomplete information, convexity and duality

Consider a two person zero sum game with incomplete information on one side defined by sets of actions $S$ and $T$, a finite parameter space $K$, a probability $p$ on $K$ and for each $k$ a real payoff function $G^k$ on $S \times T$. Assume $S$ and $T$ convex and for each $k$, $G^k$ bounded and bilinear on $S \times T$. The game is played as usual: $k \in K$ is selected according to $p$ and told to player 1 (the maximizer) while player 2 only knows $p$. In normal form, Player 1 chooses $s = \{s^k\}$ in $S^K$, Player 2 chooses $t$ in $T$ and the payoff is

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Let

\[ v(p) = \sup_{S^K} \inf_{T} G^p(s, t) \quad \text{and} \quad \overline{v}(p) = \inf_{T} \sup_{S^K} G^p(s, t) \]

then both \( \underline{v} \) and \( \overline{v} \) are concave.
Following De Meyer (1996a) one introduces, given \( z \in \mathbb{R}^k \), the “dual game” \( G^*(z) \), where player 1 chooses \( k \) and plays \( s \) in \( S \) while player 2 plays \( t \) in \( T \) and the payoff is

\[
h[z](k, s; t) = G^k(s, t) - z^k.
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Define by \( w(z) \) and \( \overline{w}(z) \) the corresponding maxmin and minmax which are convex.
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Define by $\underline{w}(z)$ and $\overline{w}(z)$ the corresponding maxmin and minmax which are convex.
Theorem

The following duality relations holds:

\[ \hat{w}(z) = \max_{p \in \Delta(K)} \left\{ \hat{v}(p) - \langle p, z \rangle \right\} \]  \hspace{1cm} (16)

\[ \hat{v}(p) = \inf_{z \in \mathbb{R}^k} \left\{ \hat{w}(z) + \langle p, z \rangle \right\} \]  \hspace{1cm} (17)

where \( \hat{f} \) stands for \( f \) or \( \overline{f} \).
Given \( z \), let \( p \) achieve the maximum in (16) and \( s \) be \( \varepsilon \)-optimal in \( G^p \): then \( (p, s) \) is \( \varepsilon \)-optimal in \( G^*(z) \).

Given \( p \), let \( z \) achieve the infimum up to \( \varepsilon \) in (17) and \( t \) be \( \varepsilon \)-optimal in \( G^*(z) \): then \( t \) is also \( 2\varepsilon \)-optimal in \( G^p \).
Given $z$, let $p$ achieve the maximum in (16) and $s$ be $\varepsilon$-optimal in $G^p$: then $(p, s)$ is $\varepsilon$-optimal in $G^*(z)$. Given $p$, let $z$ achieve the infimum up to $\varepsilon$ in (17) and $t$ be $\varepsilon$-optimal in $G^*(z)$: then $t$ is also $2\varepsilon$-optimal in $G^p$. 
6.2 The dual of a repeated game with incomplete information

We consider now repeated games with incomplete information on one side.

a) Primal and dual recursive equations

The use of the dual game will be of interest for two purposes: construction of optimal strategies for the uninformed player and asymptotic analysis.
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a) Primal and dual recursive equations

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Theorem

The maxmin satisfies in the primal game:

\[(n + 1) \nu_{n+1}(p) = \max_{x \in X^k} \min_{y \in Y} \left\{ \sum_k p^k x^k G^k y + n \sum_i \hat{x}(i) \nu_n(p(i)) \right\}\]

(18)

with \(\hat{x}(i) = \sum_k p^k x^k(i)\) and \(p^k(i) = \text{Prob}(k|i)\).

The minmax satisfies in the dual game:

\[(n + 1) \bar{w}_{n+1}(z) = \min_{y \in Y} \max_{i \in I} n \bar{w}_n \left( \frac{n + 1}{n} z - \frac{1}{n} G_i y \right)\]

(19)
Each of these ‘dynamic programming type’ equations allows to construct optimal strategies for the player operating first but the state variable is not the same.

The main advantage of dealing with (19) rather than with (18) is that the state variable evolves smoothly from $z$ to $z + \frac{1}{n}(z - G_iy)$.

Rosenberg (1998) extended the previous duality to games having at the same time incomplete information and stochastic transition on the parameters which allows to deduce properties of optimal strategies in this dual game for each player.
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b) The differential dual game

This section follows Laraki (2002) and starts again from equation (19). The recursive formula for the value $w_n$ of the dual of the $n$ stage game can be written, since $w_n(z)$ is convex, as:

$$(n+1)w_{n+1}(z) = \min_{y \in Y} \max_{x \in X} n w_n\left(\frac{n+1}{n} z - \frac{1}{n} xGy\right).$$  \hspace{1cm} (20)

This leads to consider $w_n$ as the $n^{th}$ discretization of the upper value of the differential game (of fixed duration) on $[0, 1]$ with dynamic $\zeta(t) \in \mathbb{R}^K$ given by:

$$\frac{d\zeta}{dt} = x_t Gy_t, \quad \zeta(0) = -z$$

$x_t \in X$, $y_t \in Y$ and terminal payoff $\max_k \zeta^k(1)$. 

Sylvain Sorin  Zero-Sum RG
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$x_t \in X$, $y_t \in Y$ and terminal payoff $\max_k \zeta^k(1)$. 
Basic results of the theory of differential games (see e.g. Souganidis (1999)) show that the game starting at time $t$ from state $\zeta$ has a value $\varphi(t, \zeta)$, which is the only viscosity solution, uniformly continuous in $\zeta$ uniformly in $t$, of the following partial differential equation with boundary condition:

$$\frac{\partial \varphi}{\partial t} + u(\nabla \varphi) = 0, \quad \varphi(1, \zeta) = \max_k \zeta^k. \quad (21)$$
One recovers also the speed of convergence and the identification of the limit through variational inequalities in terms of local sub- and super-differentials. Similar tools have been recently introduced by Cardaliaguet (2007, 2008, 2009) to study differential games of fixed duration and incomplete information on both sides $\Gamma(p, q)[\theta, t]$. 
7.1 Dynamic programming and MDP

In the framework of general dynamic programming (one person stochastic game with a state space $\Omega$, a correspondence $C$ from $\Omega$ to itself and a real bounded payoff $g$ on $\Omega$) Lehrer and Sorin (1992) gave an exemple where $\lim_{n \to \infty} v_n$ and $\lim_{\lambda \to 0} v_\lambda$ both exist and differ. They also proved that uniform convergence (on $\Omega$) of $v_n$ is equivalent to uniform convergence of $v_\lambda$ and then the limits are the same. However this condition alone does not imply existence of the uniform value, $v_\infty$, see Lehrer and Monderer (1994), Monderer and Sorin (1993).
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Recent advances have been obtained by Renault (2007a) introducing new notions like the values $\nu_{nm}$ (resp. $\nu_{nm}$) of the game where the payoff is the average between stage $n + 1$ and $n + m$ (resp. the minimum of all averages between stage $n + 1$ and $n + \ell$ for $\ell \leq m$).

**Theorem**

Assume that the state space $\Omega$ is metric compact and the family of functions $\nu_{nm}$ and $\nu_{nm}$ are uniformly equicontinuous. Then the uniform value $\nu_\infty$ exits.

Player 1 cannot get more than $\min_m \max_n \nu_{nm}$ and under the above condition this quantity is also $\max_n \min_m \nu_{nm}$ (and the same with $\nu$ replace by $\nu$).
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Recent advances have been obtained by Renault (2007a) introducing new notions like the values $\nu_{nm}$ (resp. $\nu_{nm}$) of the game where the payoff is the average between stage $n + 1$ and $n + m$ (resp. the minimum of all averages between stage $n + 1$ and $n + \ell$ for $\ell \leq m$).

**Theorem**

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*General MDP processes with finite state space have a uniform value*

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Consider now a game where player 1 controls the transition on the state:

basic examples are stochastic games where the transition is independent of player’s 2 moves, or games with incomplete information on one side (with no signals)

but this class also covers the case where the state is random, its evolution independent of player 2’s moves and player 1 knows more than player 2.

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Preliminaries

Evaluation of the payoffs: asymptotic and uniform approaches

Basic results

General model and further results

Recursive formula and operator approach

Incomplete information and dual game

MDP and games with one controller

Recent advances

Sylvain Sorin

Zero-Sum RG
Unification of the field: same tools for the general model including stochastic aspects, incomplete information and signals
Similar approach for games in continuous time or differential games
Compact case: recursive formula
general evaluation function
new tools: viscosity solutions and comparison arguments
link with differential games of fixed duration
example: weak approachability
study of the limit game
Uniform approach
regularity of the family of compact value functions and level of information
link with qualitative differential games
example: approachability (Spinat, 2002; Assoulamani, Quincampoix and Sorin, 2009)
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