Pricing American Options under Partial Observation of Stochastic Volatility

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Introduction and Motivation

- Stochastic volatility (SV) models capture the impact of time-varying volatility on the financial markets, and hence are heavily used in financial engineering.
- Most research on American option pricing in SV models assume that the volatility is fully observable.
- However, SV is not directly observable in reality.
- Consequence of assuming fully observable SV:
    - Overpricing of the option.
    - The optimal exercise policy not replicable in reality.

Partially Observable SV

- SV is not directly observable $\neq$ We know nothing about SV.
- SV can be inferred from the observed asset prices: a density estimator is $\mathbb{P}$(SV|history of asset prices). So SV is “partially observable”.
- This density estimator provides a full characterization of the SV based on all the available information.
- An optimal exercise policy in reality should rely on all the available information (i.e., the history of asset price).

SV Model

- Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a probability space. The SV $\{X_t\}$ and price $\{S_t\}$ of an asset follow the processes:

$$
\begin{aligned}
dX_t &= \alpha(X_t)dt + \beta(X_t)d\widetilde{W}_t^1, \\
dS_t &= S_t(rdt + \sigma(X_t)dW_t^2),
\end{aligned}
$$

  where r is the interest rate, $\{\widetilde{W}_t^1\}$ and $\{W_t^2\}$ are correlated Wiener processes with $d\widetilde{W}_t^1 dW_t^2 = \rho dt$ and $\rho \in [-1, 1]$.

- For example, in Heston Model, $\{X_t\}$ is an Ornstein-Uhlenbeck(OU) process that satisfies

$$
dX_t = \lambda(\theta - X_t)dt + \gamma d\widetilde{W}_t^1.
$$

Pricing under Partially Observable SV

- Assume a finite number of exercise opportunities $\{t_0, t_1, \ldots, t_T\}$, simply denoted as $\mathcal{J} = \{0, 1, \ldots, T\}$. Denote $\mathcal{F}_t^S \triangleq \sigma(S_0, S_1, \ldots, S_t)$.

- The option (it is in fact a Bermudan option) price is

$$V_0(s_0, \pi_0) = \max_{\tau \in \mathcal{J}, \{\mathcal{F}_t^S\}-\text{adpted}} \mathbb{E}[g_\tau(S_\tau) | S_0 = s_0, X_0 \sim \pi_0].$$

- The stopping time (exercise policy) τ only depends on the history of the asset price.

Transformation

- The above partially observable problem can be transformed to an equivalent fully observable one by introducing a new state, "filtering distribution":

$$\pi_t(x_t) = p(X_t = x_t | S_0 = s_0, \cdots, S_t = s_t), \quad t = 1, \ldots, T.$$

- Π_t(random variable form of π_t) is updated by receiving the asset price S_t at time t. Π_t satisfies the recursion

$$\Pi_t = \Phi_t(\Pi_{t-1}, S_{t-1}, S_t), \quad t = 1, \ldots, T.$$

Therefore, (S_t, Π_t) is an $\mathcal{F}_t^S$-adapted Markov process.

Theoretical Approach: Dynamic Programming

- Theoretically, the option value V_0 can be solved by dynamic programming:

$$
\begin{aligned}
V_T(s_T, \pi_T) &= g_T(s_T), \\
V_t(s_t, \pi_t) &= \max\left(g_t(s_t), C_t(s_t, \pi_t)\right), \quad t = T-1, \ldots, 1,
\end{aligned}
$$

  where the continuation value

$$
C_t(s_t, \pi_t) \triangleq \mathbb{E}[V_{t+1} | \mathcal{F}_t^S] = \mathbb{E}[V_{t+1}(S_{t+1}, \Pi_{t+1}) | S_t = s_t, X_t \sim \pi_t].
$$

- The optimal stopping time τ^* can be derived by the following recursion:

$$
\begin{aligned}
&\tau_T^* = T, \\
&\tau_t^* = \tau_{t+1}^* \cdot 1_{\{C_t(S_t, \Pi_t) > g_t(S_t)\}} + t \cdot 1_{\{C_t(S_t, \Pi_t) \le g_t(S_t)\}}, \quad t = T-1, \ldots, 1.
\end{aligned}
$$

Typical Computational Difficulties

The exact dynamic programming is impossible due to the following computational difficulties.

- The filtering distribution Π_t is infinite dimensional.
- The updating of the value function V_t or the continuation value C_t involves conditional expectations.
- (X_t, S_t) can be high dimensional.

Outline of Our Approach

An upper-and-lower-bound approach: the gap between the bounds gives an indication of the quality of the solutions.

- Asymptotic upper bound — Filtering-based duality approach
- Asymptotic lower bound — Longstaff's least square Monte Carlo method

Upper Bound: Martingale Duality

Theorem: extension of Rogers(2002), Haugh and Kogan (2004)

Let $\mathcal{M}$ represent the space of $\mathcal{F}_t^S$-adapted martingale M_t with $M_0 = 0$ and $\sup_{t\in\mathcal{J}} |M_t| < \infty$, we have

$$V_0(s_0, \pi_0) = \inf_{M\in\mathcal{M}} \left\{ \mathbb{E}[\max_{t\in\mathcal{J}}(g_t(S_t) - M_t)|S_0 = s_0, X_0 \sim \pi_0] \right\}.$$

- The process V_t is called the Snell envelop of g_t, and is a (smallest) supermartingale that dominates g_t, i.e. $\mathbb{E}[V_{t+1}|\mathcal{F}_t^S] \leq V_t$.
- The optimal martingale M_t^* is the martingale part of the Snell envelop V_t.
- By Doob-Meyer decomposition, $M_t^* = \sum_{i=0}^t \Delta_i^*$ with $\Delta_t^* = \mathbb{E}[V_t|\mathcal{F}_t^S] - \mathbb{E}[V_t|\mathcal{F}_{t-1}^S] = \mathbb{E}[g_{\tau_t^*}(S_{\tau_t^*})|\mathcal{F}_t^S] - \mathbb{E}[g_{\tau_t^*}(S_{\tau_t^*})|\mathcal{F}_{t-1}^S]$.

Upper Bound: Suboptimal Martingale

- Any $\mathcal{F}_t^S$-adapted martingale $M_t \in \mathcal{M}$ leads to an upper bound on the option price:

$$V_0(s_0, \pi_0) \leq \mathbb{E}[\max_{t \in \mathcal{J}}(g(S_t) - M_t) | S_0 = s_0, X_0 \sim \pi_0].$$

- Given a suboptimal stopping time τ, we can construct a suboptimal martingale $M_t = \sum_{i=0}^{t} \Delta_i$ with

$$\Delta_t = \mathbb{E}[g_{\tau_t}(S_{\tau_t}) | S_t, X_t \sim \Pi_t] - \mathbb{E}[g_{\tau_t}(S_{\tau_t}) | S_{t-1}, X_{t-1} \sim \Pi_{t-1}].$$

Upper Bound: The Filtering-Based Duality Approach

1. Generate N_1 independent paths of the asset price $\{s_1^{(k)}, \ldots, s_T^{(k)}\}$ with initial condition $X_0 \sim \pi_0$ and $S_0 = s_0$.
2. **For** $k = 1, 2, \cdots, N_1$, **do**
   – For $t = T, \cdots, 1$ compute $M_t^{(k)} = \Delta_1^{(k)} + \cdots + \Delta_t^{(k)}$ with
   $$\Delta_t^{(k)} = \mathbb{E}[g_{\tau_t}(S_{\tau_t})|s_t^{(k)}, \pi_t^{(k)}] - \mathbb{E}[g_{\tau_t}(S_{\tau_t})|s_{t-1}^{(k)}, \pi_{t-1}^{(k)}].$$
   – Evaluate $U^{(k)} = \max_{t\in\mathcal{J}} \left(g(S_t^{(k)}) - M_t^{(k)}\right)$. **End**
3. Set $U_{N_1}^{\tau} = \frac{1}{N_1}\sum_{k=1}^{N_1} U^{(k)}$.

$U_{N_1}^{\tau}$ is an asymptotic upper bound on the option price $V_0(s_0, \pi_0)$.

Upper Bound: Approximate Martingale Difference

- In the filtering-Based duality approach, how to compute the martingale difference?

$$\Delta_t = \mathbb{E}[g_{\tau_t}(S_{\tau_t})|s_t, \pi_t] - \mathbb{E}[g_{\tau_t}(S_{\tau_t})|s_{t-1}, \pi_{t-1}].$$

  - Particle filtering for approximating π_t: $\hat{\pi}_t = \sum_{i=1}^{m} \delta_{x_t^i}$.
  - Nested simulation for approximating $\mathbb{E}[g_{\tau_t}(S_{\tau_{t+1}})|S_t = s_t, X_t = x_t^{(i)}]$.

Upper Bound: Suboptimal Stopping Time

- In the filtering-based duality approach, how to find a suboptimal stopping time?
- We use the least square Monte Carlo method proposed by *Longstaff and Schwartz* (2001)

$$\begin{aligned}\tau_T &= T, \\ \tau_t &= \tau_{t+1}\cdot 1_{\{\tilde{C}_t(S_t)>g_t(S_t)\}} + t\cdot 1_{\{\tilde{C}_t(S_t)\le g_t(S_t)\}}, \quad t = T-1,\dots,1.\end{aligned}$$

where the approximate continuation value $\tilde{C}_t(S_t)$ is obtained by the regression method.

Asymptotic Lower Bound

- Any suboptimal stopping time τ leads to a lower bound on the option price V_0.
  - Generate N independent sample paths of the asset price $s^{(i)} = \{s_1^{(i)}, \cdots, s_T^{(i)}\}, i = 1, \ldots, N$.
  - Applying τ on all sample paths, and take the average payoff $L_N^\tau = \frac{1}{N}\sum_{i=1}^{N} g_\tau(s_\tau^{(i)})$.
- L_N^τ is an asymptotic lower bound.

Numerical Results

- We consider pricing an American put option

$$g_t(S_t) = \max\left(e^{-rt}(K - S_t), 0\right).$$

- Parameter setting:
    - Volatility parameter: $\lambda = 1$, $\theta = 0.15$,$\gamma = 0.1$, $\rho = 0$;
    - Asset price parameter: $r = 0.05$, $K = 100$;
    - Time parameter: $\delta t = 0.2, 0.1, 0.05$, and $T = 1/\delta t$;
    - Initial condition: $S_0 = 110$, $x_0 = 0.15$.
    - Basis functions:
      $h_{t1}(S_t) = \exp(S_t)$, $h_{t2}(S_t) = \exp(-S_t/2)(1 - S_t)$, $h_{t3}(S_t) = 1$.
    - Number of sample paths: $N_1 = 500, N = 40000$.

Numerical Results

Figure: Table: American Put Option Values

δt	Full Obs.	L	U (m=100)	V = (L+U)/2	Overprice	U (m=50)
0.2	1.575	1.336	1.368	1.352	0.223	1.382
0.1	1.726	1.414	1.538	1.476	0.250	1.599
0.05	1.912	1.523	1.649	1.586	0.326	1.714

- The option is overpriced about 15% if the volatility is treated as fully observable.
- Our upper and lower bound solutions are close enough, indicating that both are good approximations of the true price.
- The upper bound $U(m = 100)$ with particle number $m = 100$ is tighter than $U(m = 50)$ with particle number $m = 50$.

Conclusions

- We consider pricing American options under the realistic assumption that the SV is not directly observable.
- We propose a filtering-based duality approach, which complements a lower bound (and a suboptimal exercise policy) by an asymptotic upper bound.
- Numerical results confirm that the option is overpriced when the SV is treated as fully observable, and show that our approach provides good approximation of the true option price.

Thank you !