A Stochastic Process Approximation for Optimizing the Cost of a Production Line

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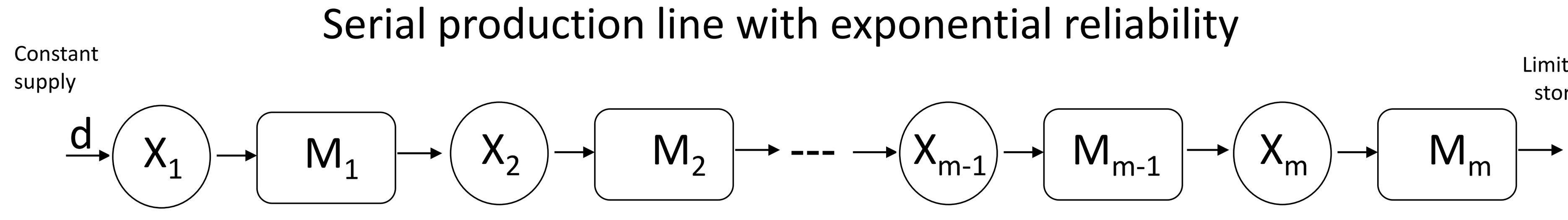
Abstract

We consider a transfer line subject to a constant rate of demand for finished parts. This constant rate of finished parts extraction can be enforced either through a pull mechanism or a push (supply rate of raw material) mechanism. Sadr and Malhamé have developed a decomposition/aggregation methodology [4,5] for the approximate performance analysis of transfer lines with rate enforcement achieved through a pull mechanism. The main advantage of this approximation scheme is that causality propagates unidirectionally (upstream to downstream), thus making it possible to use dynamic programming as an optimization tool for buffer sizing under a Kanban architecture. In this paper, we develop the dual of this decomposition/aggregation methodology, one for which the constant rate of production is enforced through a push mechanism. Causality propagates in this formulation from downstream towards upstream. The resulting approximate performance estimates (mean storage costs under a service level constraint) are validated against Monte Carlo simulations.

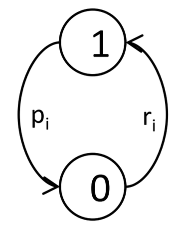
Introduction

This work aims to size buffers of a production line so that disruptions of machines do not affect a continuous flow of parts throughout the system. The methodology consists of machines' process approximation for matching the simulation results.

Serial production line with exponential reliability



Each machine, M_i, spends an exponential random time in both of their states; state 1, with parameter p_i, which is when the machine is *working properly*, and state 0, with parameter r_i, when the machine is *being repaired*.



Machine nominal production rates throughout the line are set up to be non decreasing

$$k_1 < k_2 < \cdots < k_{m+1} < k_m \quad (1)$$

This structure ensures an unidirectional propagation of production rate adjustment in the line. Real machine production rates, $u_i(t)$, behaves as follow:

$$u_i(t) = \begin{cases} u_{i-1}(t) & \text{when } X_i \text{ is empty} \\ 0 & \text{when } X_{i+1} \text{ is full} \\ k_i & \text{otherwise} \end{cases} \quad M_i \text{ is ON}; \qquad u_i(t) = 0 \quad M_i \text{ is OFF} \quad (2)$$

Furthermore, the system verifies that material is conserved throughout the line, so that the expectation of each machine production rate is equal

$$E[u_i(t)] = d_{av} \quad (3)$$

Theory

The theoritical model is built by the decomposition of the line into m single one-stock-one-machine lines. We then have the analytical equations (4), (5) and (6). Those equations are obtained by considering the system as a modified queueing model [1,2] or by considering its infenitesimal increments [3].

$$P_{o_i} = \frac{\tilde{r}_i}{\tilde{r}_i + \tilde{p}_i} \frac{1 - \mu_i}{1 - \mu_i e^{-\alpha_i z_i}} \quad (4)$$

$$P_{z_i} = \frac{\tilde{p}_i}{\tilde{r}_i + \tilde{p}_i} \frac{(1 - \mu_i) e^{-\alpha_i z_i}}{1 - \mu_i e^{-\alpha_i z_i}} \quad (5)$$

$$f_{X_i}(x) = \frac{\tilde{p}_i}{\tilde{r}_i + \tilde{p}_i} \frac{k_i}{k_i - \tilde{d}_i} \frac{\alpha_i e^{-\alpha_i x}}{1 - \mu_i e^{-\alpha_i z_i}} \quad (6)$$

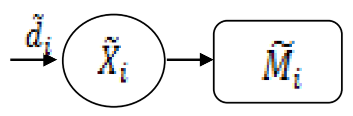


Fig.1 : One-stock-one-machine Line

$$\text{where} \quad \alpha_i = \frac{\tilde{r}_i}{\tilde{d}_i} - \frac{\tilde{p}_i}{k_i - \tilde{d}_i} \quad \text{and} \quad \mu_i = \frac{\tilde{p}_i}{\tilde{r}_i} \frac{\tilde{d}_i}{k_i - \tilde{d}_i}$$

Each machine's process in the line is approximated by a two-state Markov chain. Parameters are set up in regard to Sadr's and Malhamé's method [4,5] by matching the probability that a machine is working properly, equations (7,8).
Furthermore, applying the average principle to the inventories' supply, the income stochastic process can be replaced by its average (equation (9)).

$$P(M_i \text{ is working}) = \frac{r_i}{r_i + p_i}\left(1 - P_{z_{i+1}}\right) = \frac{\tilde{r}_i}{\tilde{r}_i + \tilde{p}_i} \quad (7)$$

$$\tilde{r}_i = r_i \quad (8a)$$

$$\tilde{p}_i = \frac{r_i + p_i}{b_{i+1}} - r_i \quad (8b)$$

$$\tilde{d}_i = \frac{d_{av}}{b_i} \quad (9)$$

$$b_i = 1 - P_{z_i} \quad (10)$$

Thus, the equations (4,5,6) are validated to compute the expectation of each of the inventories in the line.

The global cost of holding the inventories in the line can then be obtained by computing the sum of the expectation of the different inventories by their individual linear price c_i , equation (14).

$$\sum_{i=1}^{m} c_i E[X_i] = \sum_{i=1}^{m} c_i \left(\int_0^{z_i} x f_{X_i}(x) dx + z_i P_{z_i}\right) = \sum_{i=1}^{m} T_i(b_i, b_{i+1}) \quad (14)$$

$$\forall i \in [1,m] \quad b_i \in \left]\frac{d_{av}}{k_i}, 1\right[\quad b_{i+1} \in A_{i+1}(b_i)$$

$$A_{i+1}(b_i) = \left\{(b_i, b_i) \in [0,1]^2 \ \ s.t. \ \ b_i > \frac{r_i}{r_i + p_i} b_{i+1} \ \text{and} \ \frac{r_i}{r_i + p_i} b_{i+1} k_i > d_{av}\right\} \quad (15)$$

$$T_i(b_i, b_{i+1}) = \frac{c_i}{\alpha_i} \frac{\tilde{d}_i}{k_i - \tilde{d}_i}\left(\frac{k_i}{\tilde{d}_i}\left(b_i - \frac{\tilde{r}_i}{\tilde{p}_i + \tilde{r}_i}\right) - \frac{1 - b_i}{1 - \mu_i} \frac{\tilde{p}_i + \tilde{r}_i}{\tilde{r}_i} \ln\left(\frac{\mu_i}{1 - b_i}\left(\frac{\tilde{r}_i}{\tilde{p}_i + \tilde{r}_i} \frac{k_i}{\tilde{d}_i} - b_i\right)\right)\right) \quad (16)$$

In addition to the model, we consider the case for which the supply at the beginning of the line is never disrupted. It is achieved by adding an infinite capacity of storage next to the system, which can be an outsourced capacity, for example. Its linear price is set to be $c_{1'}$.

$$T_1(b_1 = 1, b_2) = \frac{c_1}{\alpha_1}\left(1 + \ln\left(\frac{c_{1'}}{c_1} \frac{\tilde{p}_1}{\tilde{p}_1 + \tilde{r}_1} \frac{k_1}{k_1 - d}\right)\right) \quad (17)$$

Our objective is then to reach the following minimum:

$$J_{tot} = \min_{(b_i)_{i \in [1,m]}} \left\{\sum_{i=1}^{m} T_i(b_i, b_{i+1})\right\} \quad (18)$$

Results

For the following results, machines have been chosen with equal parameters r=.9 and p=.1. The difference between two consecutive production rates is of .2 unity, the expecting average production rate is set up to be 1. Lastly, linear inventory costs equal 1.

The following table shows that up to a 5 machines line the simulation validates the theoritical model with a maximum difference of 5.796%. For a 10 machines line, a 1.356% difference per inventory makes the model interesting.

Machines #	1	2	3	4	5	…	10
Simulated Global Cost	0.0699	0.2179	0.3693	0.5207	0.6798		1.5135
Difference	-0.57%	-0.78%	-2.33%	-3.61%	-5.796%		-13.56%
Theoretical Global Cost	0.0695	0.2162	0.3607	0.5019	0.6404		1.3083

In figure 4, the global cost of the line $J_{tot} = J_1(b_1)$ is non decreasingm and the variable b_1 represents the line's efficiency. It shows how the global cost is linked to the line's efficiency. In figure 5, the optimal inventories are drawn for a 95% line efficiency. The curve is smooth and starts high to ensure the 95% line efficiency. It then decreases and stabilizes to a lower level showing that lower inventories can ensure the expected average rate of production dav.

Fig 4: Global Cost of the System

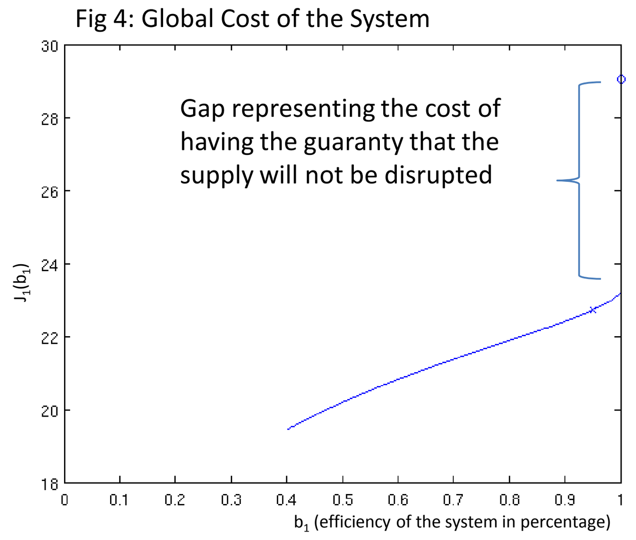
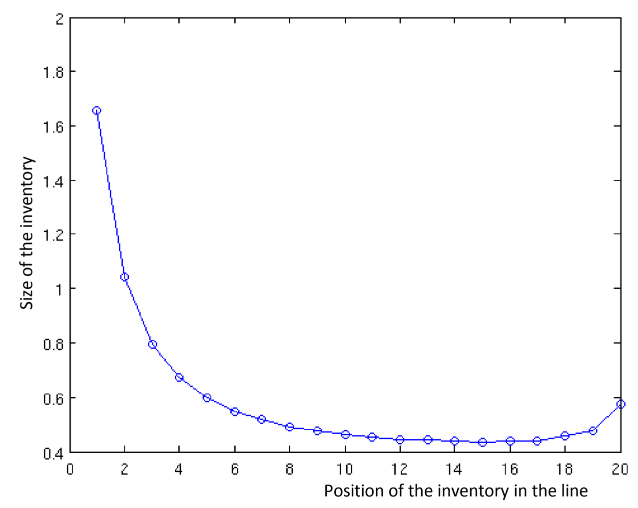


Fig 5: Optimal Inventories' Size in the Line



Dynamic Programming Formulation

The minimum (18) can be found using dynamic programming. The dependency for each stock X_i on b_i and b_{i+1} only gives the opportunity to compute the minimum expectation of the inventory cost from stock i to m by a recursive method. We then set up the Bellman equations for i=1..m-1 :

$$g(X_i, b_i, b_{i+1}) = c_i X_i \quad (19a)$$

$$J_m(b_m) = E[g(X_m, b_m, b_{m+1})] = T_m(b_m, b_{m+1}) \quad (19b)$$

$$\begin{aligned} J_i(b_i) &= \min_{b_{i+1}} E[g(X_i, b_i, b_{i+1}) + J_{i+1}(b_{i+1})] \\ &= \min_{b_{i+1} \in A_{i+1}(b_i)} E[T_i(b_i, b_{i+1}) + J_{i+1}(b_{i+1})] \end{aligned} \quad (19c)$$

$$J_{tot}(b_1) = \min_{(b_i)_{i \in [2,m]}} \left\{\sum_{i=1}^{m} T_i(b_i, b_{i+1})\right\} = J_1(b_1) \quad (20)$$

Pseudo-Algorithm of Dynamic Programming

```
Compute J_m (b_m); l=m-1;

while l>0;
  for i > d_av/k(l)

    for j in A(i)
      Compute T_l (i,j)+J_l+1 (j);
    end

    J_l (i) = min (T_l(i,j)+J_l+1 (j));
  end

  l = l-1;
end
```

Simulation Formulation

A Monte Carlo simulation is used to obtain the expected global cost of holding the inventories in the line. The following two main steps describe the system's behavior and the event-driven simulation that is programmed:

-First, the machines' random behaviors are generated to define the simulation's horizon.
-Secondly, the inventories' evolution becomes deterministic with the given machines' horizon and it can be drawn by considering production rates' changes over the simulation.

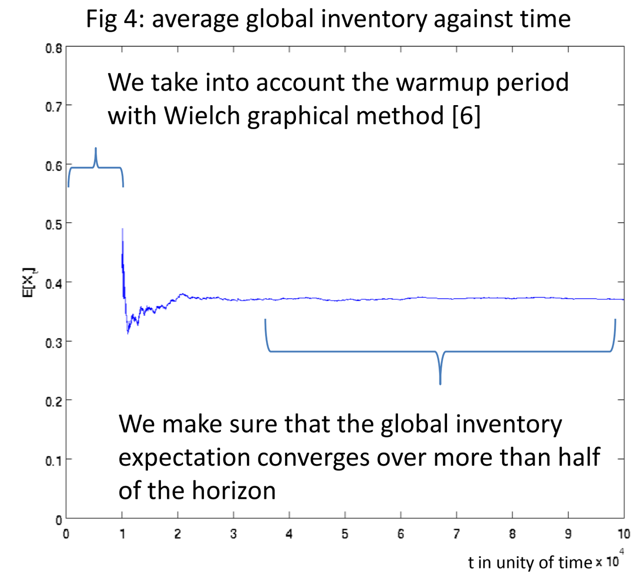
The simulation has been written with Matlab2010b and the algorithm is summarized in the following pseudo-algorithm:

Pseudo-algorithm of Simulation

```
Read Data, m; K; Z; T;

Generate a random horizon for each
machine

Initialize Data, X=Z/2; t=0; j=1; U=K;

while t(j) < T
  Detect next event, E;
  Pickup its time, t';

  t(j+1) = t(j) + t';
  X(j+1) = X(j) + U*t+;

  Update variable, U;

  j=j+1;
end
```

The resulting output of the simulation is shown in figure 4. The global cost is obtained for a 3 machines line with the same parameters described in the results section.

Fig 4: average global inventory against time



Discussion

This approach provides results for production lines with rates of production that are non-decreasing and exponential random times of machines being repaired and working properly. This extension of the method developed by Sadr and Malhame [4,5] with a constant demand led to a maximum 5% inaccuracy for line up to 5 machines. For lines with more machines, the inaccuracy becomes bigger, but an almost 1% inaccuracy per inventory makes the model interesting for further application and development.

The model could be improved by considering a factor that would reduce the inaccuracy per inventory in the line. For instance, reducing parameters of machines in the line adequately would be possible for matching simulation results.

Finally, other works could be added to this one. It would be possible to consider a minimum service rate at the end of the line for matching a specific demand.

It would also be possible to consider different policies for driving the line production and better schedule delivery of materials to customers. For instance, Mhada and Malhamé [7] have considered an extension of Sadr and Malhamé works [4,5] for which conwip discipline is considered. A similar work could be applied to this one.

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