A New Perspective on Feasibility Determination

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Motivation

- We consider the problem of feasibility determination in a stochastic setting.
- We wish to determine whether a system belongs to a set $\Gamma \in \Re^d$ based on a performance measure estimated through Monte Carlo simulation.

Literature Review

- Ranking and Selection
  - Optimal Computing Budget Allocation
  - Value of Information
- Multiple Performance Metrics
  - Gupta, Nagel, and Panchapakesan (1973)
  - Santner and Tamhane (1984)
  - Andijani (1998)
  - Butler, Morrice, and Mullarkey (2001)
  - Lee, Chew, and Tang (2006, 2007)

Literature Review

- Stochastic Constraints
  - Andradóttir, Goldsman and Kim (2005)
  - Batur and Kim (2005)
  - Batur and Kim (2010)
  - Hunter and Paupathy (2010)

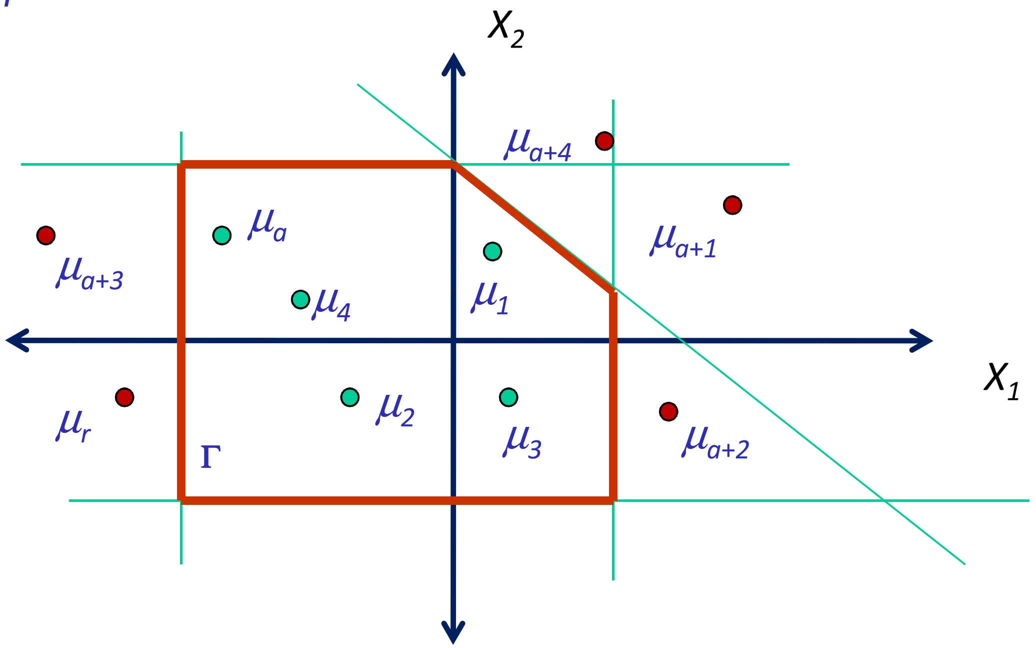
Literature Review

- Our approach
  - Large deviations theory
  - Glynn and Juneja (2004)
  - Hunter and Pasupathy (2010)

- Our contribution
  - Characterize optimal allocation where the feasible set is defined through linear constraints
  - Provide an algorithm to achieve optimal allocation

Problem Definition

- Consider r systems with unknown performance measures: $\mu_1, \ldots, \mu_r$ where μ_i = E[X_i] for some random vector $X_i \in \Re^d$
- Let $\Gamma = \{x \in R^d : Ax \leq b\}$
- Without loss of generality, let us assume that $\mu_1, \ldots, \mu_a \in \Gamma$ and $\mu_{a+1}, \ldots, \mu_r \notin \Gamma$.



Problem Definition

- The feasibility determination problem can then be defined as:

$$\min_{p_1,\ldots,p_r \in \mathcal{M}} g_n(p_1,\ldots,p_r)$$

where

$$g_n(p_1,\ldots,p_r) = \sum_{i=1}^{a} P(\bar{X}_i(p_i n) \notin \Gamma) + \sum_{i=a+1}^{r} P(\bar{X}_i(p_i n) \in \Gamma)$$

- p_i's represent the fraction of the simulation budget that is allocated to sampling from system *i*, where for simplicity we assume that each system has the same per-sample cost.
- The simulation budget, *n*, is allocated in order to minimize the expected number of incorrect classifications.

Notation and Assumptions

- Let $I_i(\bullet)$ be the large deviations rate function:

$$I_i(x) = \sup_{\theta \in R^d} \{\theta \cdot x - \log M_i(\theta)\},$$

where $$M_i(\theta) = E \exp(\theta \cdot X_i)$$

- We assume that:
  - A1: The performance measures do not exactly lie at the boundary, i.e., $\mu_i \notin \partial\Gamma$ for all *i.*
  - A2: For each system, there exists $\theta_i^* \in D_i^o$ such that $(\log(M_i(\theta_i^*))' = \gamma_i, \gamma_i \in \partial\Gamma,$ is a point where the large deviations rate function is minimal over $\partial\Gamma$.
  - A3: The underlying distributions of the performance measures have bounded support.

Results

Proposition 1. *Suppose Assumptions A1 and A2 hold. Then*

$$\lim_{n\to\infty} \frac{1}{n} \log g_n(p_1, \ldots, p_r) = -\min_i p_i I_i(\gamma_i).$$

Proposition 2. *Suppose assumptions A1 and A2 hold. Then*

$$\lim_{n\to\infty} \frac{1}{n} \log g_n(p_1^*, \ldots, p_r^*) = -\frac{1}{\sum_{j=1}^r I_j^{-1}(\gamma_j)}$$

where

$$p_i^* = \frac{I_i^{-1}(\gamma_i)}{\sum_{j=1}^r I_j^{-1}(\gamma_j)}.$$

For any other $(p_1, \ldots, p_r) \in \mathcal{M}$ *we have*

$$\liminf_{n\to\infty} \frac{1}{n} \log g_n(p_1, \ldots, p_r) \geq -\frac{1}{\sum_{j=1}^r I_j^{-1}(\gamma_j)}.$$

Motivation for Adaptive Approach

Issue: The fractional allocations depend on the knowledge about the location of the performance measure, μ_i, which is precisely what we are trying to determine.

Approach:

- Estimate $I_i(\gamma)$ at stage n by $I_{i,n}$
- Determine where to look next by sampling from the probability mass function $p_i^* = \frac{I_i^{-1}(\gamma_i)}{\sum_{j=1}^{r} I_j^{-1}(\gamma_j)}$

Key:

- Update twisting parameters $\theta_{i,n}$ so that $\dfrac{\frac{1}{n}\sum_{k=1}^{n} X_{i,n}\exp(\theta_{i,n}X_{i,n})}{\frac{1}{n}\sum_{k=1}^{n}\exp(\theta_{i,n}X_{i,n})} \to \gamma.$
- Hence, $I_{i,n} \to I_i(\gamma), n \to \infty.$

Constraint Set

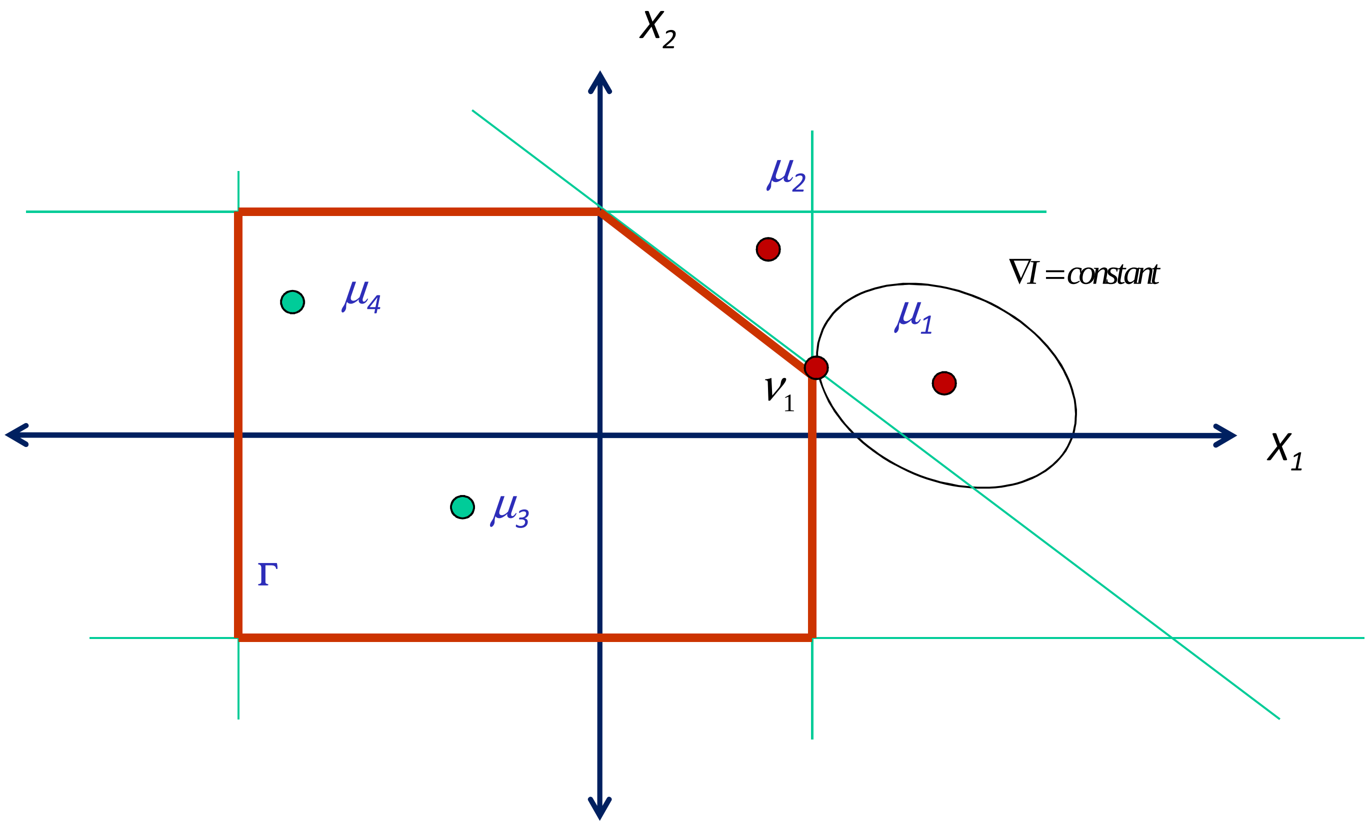
The linearity assumption is key.

For systems with $\mu_i \in \Gamma$, we can treat each constraint separately

$\Rightarrow$ The feasibility determination problem then decomposes into *d* single-dimensional problems

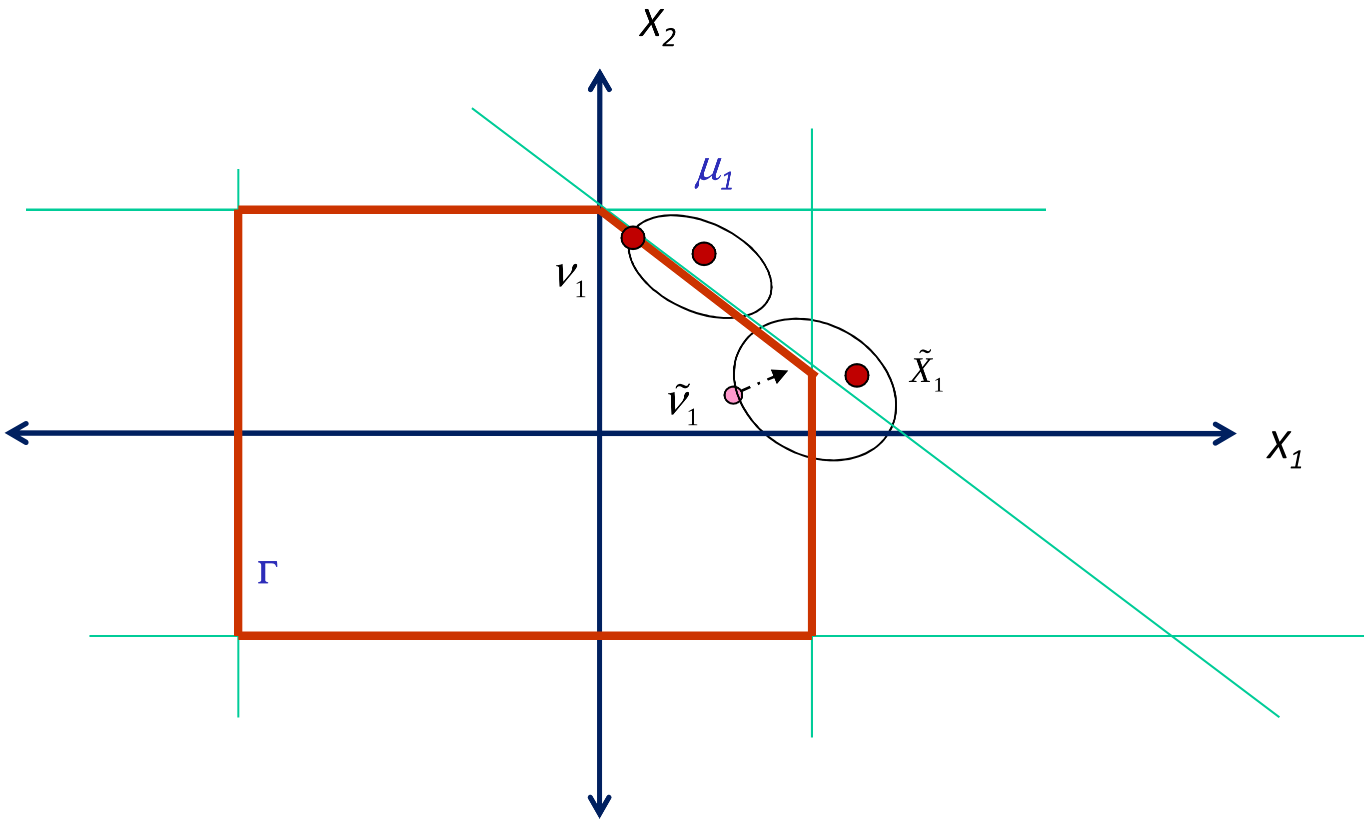
For systems with $\mu_i \notin \Gamma$, then, since Γ is convex, there exists a unique dominating point $\gamma_i \in \partial\Gamma$, where I_i achieves its minimum over Γ and the standard approach works.

Updating the Dominating Point Estimator



Updating the Dominating Point Estimator

This is the case where the DP estimator is in Γ while the sample average is outside of Γ.
We push ν in the direction of the negative gradient of the rate function while maintaining feasibility.



Key Results

Fractional allocations generated by the SA algorithm approach optimal allocations:

$$\frac{\lambda_{i,n}}{n} \to p_i^* \qquad \text{w.p. 1 as } n \to \infty$$

Expected number of incorrect determinations decays at fastest possible rate:

$$\frac{1}{n} \log \left(\sum_{i=1}^{a} P(\tilde{X}_{i,n} \notin \Gamma) + \sum_{i=a+1}^{r} P(\tilde{X}_{i,n} \in \Gamma) \right) \to -\frac{1}{\sum_{j=1}^{r} I_j^{-1}(\gamma)} \qquad \text{as } n \to \infty$$

Summary

Our contribution:

- Characterize optimal allocation of computing budget
- Provide an algorithm that leads to optimal allocation
- Prove validity for all distributions with either finite support

Current Work:

- Relax the linearity assumption in defining the feasible region